A second type of limit

We will now investigate the types of limit you have encountered in algebra when considering vertical asymptotes. These new types of limits will be defined in terms of limits at infinity.

1 Righthand limits

Suppose that the domain of the function f contains the interval (b,c). For example, $f(x) = \sin(x)/\sqrt{x(1-x)}$ has as its domain $(0,1)$. We want to make precise the behavior of f as x approaches b from above.

If L is a real number, we say that L **is the limit of** $f(x)$ **as** x **approaches** b **from above**, written

$$\lim_{y\to b^+} f(y) = L,$$

if

$$\lim_{x\to\infty} f\left(b + \frac{1}{x}\right) = L.$$

In particular we will say that

$$\lim_{h\to 0^+} f(a+h) = L$$

if

$$\lim_{x\to\infty} f(a + (1/x)) = L.$$

For example, since we know that

$$\lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = 1$$

we have

$$\lim_{y\to 0^+} \frac{\sin(y)}{y} = 1$$

and since we know that

$$\lim_{x\to\infty} \frac{1/x}{1+(1/x)} = 1$$

we know that

$$\lim_{y\to 0^+} \frac{y}{1+y} = 1.$$

Moreover, we have

$$\lim_{h\to 0^+} f(a+h) = L$$

if and only if

$$\lim_{y\to a^+} f(y) = L.$$

We have the following theorem.

Theorem 1 *Suppose that the domain of f contains the interval (b,c). Then*

$$\lim_{y\to a^+} f(y) = L$$

is equivalent to knowing that for every $t > 0$ the is some $d_t > 0$ so that if $b < y < b + d_t$ then $|f(y) - L| < t$.

The reason is that if we can find $F_t > 0$ so that $x > F_t$ implies that

$$\begin{aligned} a + (1/x) &< c \\ |f(a + (1/x)) - L| &< t \end{aligned}$$

then by setting $d_t = 1/F_t$, and $x = 1/(y-a)$ we have

$$\begin{aligned} 0 &< y - a < d_t \\ x &> F_t \end{aligned}$$

and

$$|f(y) - L| = |f(a + (y-a)) - L| = \left| f\left(a + \frac{1}{x}\right) - L \right| < t.$$

On the other hand, if we know that for some if $a < y < a + d_t$ implies $|f(y) - L| < t$ then for $x > 1/d_t$ we have $a < a + (1/x) < a + d_t$ and

$$\left| f\left(a + \frac{1}{x}\right) - L \right| < t. \qquad \textbf{QED}$$

It is vital to note that b is not required to be in the domain of f. For example, if

$$f(x) = \frac{x^2}{x}$$

0 is not in the domain of f, but it is easy to see that

$$\lim_{x \to 0^+} f(x) = 0$$

since $f(x) = x$ if $x \neq 0$ and so we may always take $d_t = t$ in the limit definition.

2 Lefthand limits

In a similar fashion we define lefthand limits. Suppose that the domain of the function f contains the interval (a, b). If L is a real number, we say that L **is the limit of $f(y)$ as y approaches b from below**, written

$$\lim_{y \to b^-} f(y) = L,$$

if

$$\lim_{x \to \infty} f\left(a - \frac{1}{x}\right) = L.$$

3 Limits

Finally, we define general limits. Suppose that the domain of the function f contains the set $(a, b) \cup (b, c)$. If L is a real number, we say that L **is the limit of $f(y)$ as y approaches b**, written

$$\lim_{\to b} f(y) = L,$$

if both of the following are true:

$$\begin{aligned} \lim_{y \to b^+} f(y) &= L \\ \lim_{y \to b^-} f(y) &= L \end{aligned}$$

We have the following theorem whose proof is similar to that of Theorem 1:

Theorem 2

$$\lim_{y \to b} f(y) = L$$

is equivalent to saying that for every $t > 0$ *the is some* $d_t > 0$ *so that if* $0 < |x - b| < d_t$ *then* $|f(x) - L| < t$.

4 Limit Theorems

The following theorems are all consequences of and parallel to theorems for limits at infinity. Except as noted, lefthand and righthand limits can be substituted for the more general limit. Also, all the functions are assumed to have a common domain containing the set $(a, b) \cup (b, c)$ for $a < b < c$.

Theorem 3 (Translation to 0)

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{h \to 0} f(a + h) = L$$

Theorem 4 (Monotone Bounded Functions) *If f is monotone and bounded then there are numbers L^- and L^+ so that*

$$\begin{aligned} \lim_{y \to b^-} f(y) &= L^- \\ \lim_{y \to b^+} f(y) &= L^+ \end{aligned}$$

Theorem 5 (Shift and Scale) *If A and B are real numbers, $m > 0$, and*

$$\lim_{y \to b} f(y) = L$$

then

$$\begin{aligned} \lim_{y \to b} Af(y) + B &= AL + B \\ \lim_{y \to b} f(my) &= L \\ \lim_{h \to 0} f(b + h) &= L \end{aligned}$$

For the Shift and Scale Theorem to hold for one sided limits we need $m > 0$.

Theorem 6 (Pinching Principle) *If $f(x) \leq g(x) \leq h(x)$ and*

$$\begin{aligned} \lim_{x \to b} f(x) &= L \\ \lim_{x \to b} h(x) &= L \end{aligned}$$

then

$$\lim_{x \to b} g(x) = L.$$

Theorem 7 (Arithmetic of Limits) *Suppose that*

$$\begin{aligned} \lim_{x \to b} f(x) &= F \\ \lim_{x \to b} g(x) &= G \\ \lim_{x \to b} h(x) &= H \neq 0. \end{aligned}$$

Then

$$\begin{aligned} \lim_{x \to b} f(x) + g(x) &= F + G \\ \lim_{x \to b} f(x)g(x) &= FG \\ \lim_{x \to b} f(x)/g(x) &= F/H \end{aligned}$$

In particular, if $R(x)$ is rational function and $R(b) = B$ then

$$\lim_{x \to b} R(x) = B.$$

We have the following definition. Suppose that $a < b < c$. A function f defined on (a, c) is said to be **continuous at** b if

$$\lim_{x \to b} f(x) = f(b).$$

This may also be expressed by saying that

$$\lim_{h \to 0} f(b + h) = f(b).$$

f is said to be **continuous** if it is continuous at each point in its domain. Every rational function is continuous. If the domain of a function is an interval containing an endpoint a, then we require the appropriate one sided limit to equal $f(a)$ to say that f is continuous at a. For example, we shall show that $f(x) = \sqrt{x(1 - x)}$ is continuous on $[0, 1]$.

A function is said to be **discontinous** at b if it is not continuous at b, and b is said to be a **point of discontinuity**. Not all points of discontinuity are the same. For example, it may be that $f(x)$ has limit, L, as x approaches b, but either this limit is not $f(b)$ or $f(b)$ is not defined at all. Such a discontinuity is said to be **removable**, for if we redefine f to give $f(b) = L$, then f is continuous at b. A prime example is $f(x) = \sin(x)/x$. If we declare that f should have the value 1 at 0, then f is continuous at $x = 0$.