1 The Natural Logarithm Function

The natural logarithm function, ln, is the inverse of the natural exponential function. As such, its domain is the positive real numbers and its range is all real numbers. You have studied its algebraic properties in College Algebra. These properties include:

$$\begin{aligned}
\ln(\exp(x)) &= x \\
\exp(\ln(x)) &= x, \text{ for } x > 0 \\
\ln(ab) &= \ln(a) + \ln(b) \\
\ln(a^r) &= r\ln(a) \\
\ln(1) &= 0 \\
\ln(\mathrm{e}) &= 1
\end{aligned}$$

where $a > 0$, $b > 0$ and r is any rational number. Since exp is strictly increasing, so is ln.

Recall that if $x < 1$ then $\exp(x) \leq 1/(1-x)$.

If we have $r > -1$ then $r/(r+1) < 1$, so it follows for $r > -1$ that

$$\exp\left(\frac{r}{r+1}\right) \leq 1 + r \tag{1}$$

It then follows from (1) that for $r > -1$ that

$$\frac{r}{r+1} \leq \ln(1+r). \tag{2}$$

On the other hand, we know that for all real numbers x that $1 + x \leq \exp(x)$ so for $r > -1$

$$\ln(1+r) \leq r. \tag{3}$$

Altogether, for $x > -1$

$$\frac{x}{x+1} \leq \ln(1+x) \leq x \tag{4}$$

The estimate (4) is handier than you might think, as there are many opportunities to apply it with $|x| < 1$.

2 The Birthday Problem

Suppose that there are N people in a room, where $1 < N < 366$. We want to find the value of N so that the chances that at least two people are born on the same day are about 1/2. We will pretend that there is no leap year. In that case if we assume that all combinations of birthdays are equally likely, we see that the probability of no two people being born on the same day, q_N is given by

$$Q_N = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \cdots \frac{365-(N-1)}{365}$$

We can rewrite this as

$$Q_N = \left(1 - \frac{1}{365}\right) \times \left(1 - \frac{2}{365}\right) \times \cdots \times \left(1 - \frac{N-1}{365}\right).$$

Therefore

$$\begin{aligned}
\ln(Q_N) &= \sum_{k=1}^{N-1} \ln\left(1 - \frac{k}{365}\right) \\
&\leq -\sum_{k=1}^{N-1} \frac{k}{365} \\
&= -\frac{(N-1)N}{730}
\end{aligned}$$

If we assume that the inquality is an equality we get that $N = 23$. If this assumption makes us nervous, we also note that since $-1 < x < 0$ implies that $x/(1+x) > x - x^2$ we also have

$$\begin{aligned}
\ln(Q_N) &= \sum_{k=1}^{N-1} \ln\left(1 - \frac{k}{365}\right) \\
&\geq -\sum_{k=1}^{N-1} \frac{k}{365} - \sum_{k=1}^{N-1} \frac{k^2}{365^2} \\
&= -\frac{(N-1)N}{730} - \frac{N(2N-1)(N-1)}{799350}
\end{aligned}$$

and this latter expression also is about equal to $\ln(1/2)$ at $N = 23$. Direct calculation reveals that $Q_{23} \approx .4927027657$.