

The natural exponential function

In this section we will prove the following theorem.

Theorem 1 *There is a strictly increasing function $\exp : (-\infty, \infty) \rightarrow (0, \infty)$ such that*

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = \exp(x).$$

In particular, $\exp(0) = 1$, and the range of \exp is $(0, \infty)$.

Furthermore, for every pair of real numbers r and s we have $\exp(r)\exp(s) = \exp(r+s)$. Lastly we will show that for every real number x ,

$$1 + x \leq \exp(x) \tag{1}$$

while if $x < 1$ then

$$\exp(x) \leq \frac{1}{1-x} \tag{2}$$

1 Defining the natural exponential function

We have established that for $r \geq 0$ that

$$E_n(r) := \left(1 + \frac{r}{n}\right)^n$$

has a limit as n tends to infinity. We denote this limit by $\exp(r)$ and we already know that

$$E_n(r) \leq \exp(r) \text{ for } r \geq 0 \tag{3}$$

$$1 + r \leq \exp(r) \text{ for } r \geq 0 \tag{4}$$

$$\exp(r) \leq \frac{1}{1-r} \text{ for } 0 \leq r < 1 \tag{5}$$

We now want to establish that $E_n(r)$ has a limit for any real number r as n tends to infinity. We will denote this limit as $\exp(r)$ and show that $\exp(r+s) = \exp(r)\exp(s)$.

Lemma 1 *Let a and b be real numbers. Then*

$$\left| \left(1 + \frac{a}{n} + \frac{b}{n^2}\right)^n - \left(1 + \frac{a}{n}\right)^n \right| \leq \frac{\exp(|a| + |b|)}{n}. \tag{6}$$

Reason: We use the binomial theorem.

$$\begin{aligned} \left| \left(1 + \frac{a}{n} + \frac{b}{n^2}\right)^n - \left(1 + \frac{a}{n}\right)^n \right| &= \left| \sum_{k=1}^n \binom{n}{k} \left(\frac{b}{n^2}\right)^k \left(1 + \frac{a}{n}\right)^{n-k} \right| \\ &\leq \sum_{k=1}^n \binom{n}{k} \left(\frac{|b|}{n^2}\right)^k \left(1 + \frac{|a|}{n}\right)^{n-k} \\ &\leq \frac{1}{n} \sum_{k=1}^n \binom{n}{k} \left(\frac{|b|}{n}\right)^k \left(1 + \frac{|a|}{n}\right)^{n-k} \\ &\leq \frac{1}{n} \sum_{k=0}^n \binom{n}{k} \left(\frac{|b|}{n}\right)^k \left(1 + \frac{|a|}{n}\right)^{n-k} \\ &= \frac{1}{n} \left(1 + \frac{|a| + |b|}{n}\right)^n \\ &\leq \frac{\exp(|a| + |b|)}{n} \end{aligned}$$

as claimed.

QED

Theorem 2 *For each real number r the sequence $E_n(r)$ is convergent to a limit that we denote by $\exp(r)$. Furthermore, if r and s are real numbers then $\exp(r+s) = \exp(r)\exp(s)$.*

Proof: We already know that if $r \geq 0$ then there is a number $\exp(r)$ so that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n = \exp(r).$$

Observe that

$$\left(1 + \frac{r}{n}\right)^n \left(1 - \frac{r}{n}\right)^n = \left(1 - \frac{r^2}{n^2}\right)^n,$$

so it follows from Lemma 1 with $a = 0$ and $b = r^2$ that

$$|E_n(r)E_n(-r) - 1| \leq \frac{\exp(r^2)}{n}$$

Applying the Principle we conclude that

$$\lim_{n \rightarrow \infty} E_n(r)E_n(-r) = 1.$$

Suppose then that $r > 0$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} E_n(-r) &= \lim_{n \rightarrow \infty} \left(E_n(r)E_n(-r) \frac{1}{E_n(r)} \right) \\ &= \left(\lim_{n \rightarrow \infty} E_n(r)E_n(-r) \right) \times \left(\lim_{n \rightarrow \infty} \frac{1}{E_n(r)} \right) \\ &= 1 \times \frac{1}{\exp(r)} = \frac{1}{\exp(r)} \end{aligned}$$

Hence $E_n(x)$ converges as n tends to infinity even when $x < 0$, so $\exp(x)$ is defined for all real numbers, and

$$\exp(x) \exp(-x) = 1.$$

Next, suppose that r and s are any two real numbers. Observe that

$$\left(1 + \frac{r}{n}\right) \left(1 + \frac{s}{n}\right) = \left(1 + \frac{r+s}{n} + \frac{rs}{n^2}\right).$$

Applying Lemma 1 with $a = r + s$ and $b = rs$ along with the pinching theorem we see that

$$\lim_{n \rightarrow \infty} (E_n(r)E_n(s) - E_n(r+s)) = 0.$$

On the other hand, we know that

$$\lim_{n \rightarrow \infty} (E_n(r)E_n(s) - E_n(r+s)) = \exp(r) \exp(s) - \exp(r+s)$$

so

$$\exp(r+s) = \exp(r) \exp(s).$$

In other words, \exp is an exponential function! We make the following definition. The **natural exponential function**, denote by \exp , has domain equal to all the real numbers, and for each real number x ,

$$\exp(x) = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n.$$

Here are some important properties of \exp .

Theorem 3

$$\begin{aligned} \exp(x) \exp(y) &= \exp(x+y) \\ \exp(0) &= 1 \\ \exp(rx) &= (\exp(x))^r \end{aligned}$$

if r is a rational number.

Reason: The last claim needs some explanation. It is certainly true if r is a non-negative integer. If r is a negative integer then

$$\exp(rx) = \frac{1}{\exp(-rx)} = \frac{1}{\exp(x)^{-r}} = \frac{1}{(\exp(x))^{-r}} = (\exp(x))^r.$$

Now suppose that q is a positive integer. Then

$$\exp(x) = \exp(qx/q) = (\exp(x/q))^q$$

so $\exp(x/q) = \sqrt[q]{\exp(x)} = (\exp(x))^{1/q}$.

So if $r = p/q$ and p and $q > 0$ are integers then

$$\exp(rx) = \exp(px/q) = (\exp(px))^{1/q} = (\exp(x))^{p/q} = (\exp(x))^r.$$

Recall now that the number e is defined to be $\exp(1)$.

Corollary 1 *If r is a rational number then $\exp(r) = e^r$.*

In light of this corollary, the expression e^x is taken to mean $\exp(x)$ for every real number. For example $e^{\sqrt{2}}$ means $\exp(\sqrt{2})$.

It remains to show that the range of the natural exponential function, \exp , is the set of positive real numbers. Observe first that \exp is strictly increasing, for if $r < s$ then

$$\exp(s) - \exp(r) = \exp(r)(\exp(s-r) - 1) > \exp(r)(1 + (s-r) - 1) = \exp(r)(s-r) > 0.$$

Next, since $\exp(1) > 2$, we know that for any positive integer n , $\exp(n) > 2^n$ so the range of \exp is not bounded above. Furthermore, since $\exp(-1) = 1/\exp(1)$ we know that $\exp(-1) < (1/2)$. Hence for any positive integer n , $0 < \exp(-n) < (1/2)^n$. This tells us that the greatest lower bound of the range of \exp is 0. We now need to show that the range contains every positive real number. Let $c > 0$ be given. Put

$$\begin{aligned} A &= \{x : \exp(x) < c\} \\ B &= \{x : \exp(x) > c\} \end{aligned}$$

Since \exp is strictly increasing there is a real number r so that $A \subset (-\infty, r]$ and $B \subset [r, \infty)$. Let

$$\begin{aligned} a &= \lim_{x \rightarrow \infty} \exp(r - x^{-1}) \\ b &= \lim_{x \rightarrow \infty} \exp(r + x^{-1}) \end{aligned}$$

We have $a \leq c \leq b$ and $a \leq \exp(r) \leq b$. If we show that $a = b$ then we have shown that $c = \exp(r)$, so that c is in the range of \exp .

Recall that if $0 < s < 1$ then

$$\exp(s) - 1 \leq \frac{1}{1-s} - 1 = \frac{s}{1-s}.$$

Observe that if $x > 1$ then

$$\begin{aligned} \exp(r) - \exp(r - x^{-1}) &\leq \exp(r - x^{-1})(\exp(1/x) - 1) \\ &< \exp(r) \frac{1}{x-1}, \end{aligned}$$

and

$$\begin{aligned} \exp(r + x^{-1}) - \exp(r) &= \exp(r)(\exp(1/x) - 1) \\ &\leq \exp(r) \frac{1}{x-1}. \end{aligned}$$

Combining these inequalities we have

$$\begin{aligned} 0 \leq b - a &\leq \exp(r + x^{-1}) - \exp(r - x^{-1}) \\ &= (\exp(r + x^{-1}) - \exp(r)) + (\exp(r) - \exp(r - x^{-1})) \\ &\leq \frac{2\exp(r)}{x-1} \end{aligned}$$

so it follows from the Pinching Principle that $a = b$.

Finally, we need to establish (1) and (2).

Recall that if $r \geq 0$ then $1 + r \leq \exp(r)$. Therefore for $r \geq 0$ we have

$$\exp(-r) \leq \frac{1}{1+r}$$

Letting $s = -r$ we have that for $s < 0$:

$$\exp(s) \leq \frac{1}{1-s}.$$

Since we also know that for $0 \leq x < 1$ that $\exp(x) \leq 1/(1-x)$ we have established (2).

It follows then from (2) that for $x < 1$ that $\exp(-x) \geq 1 - x$, so for any $s < 0$ we have $\exp(s) \geq 1 + s$. Therefore (1) holds.