Asymptotes

If f and g are functions with a common domain that is unbounded above, then we say the graphs $y = f(x)$ and $y = g(x)$ are asymptotic if

$$\lim_{x\to\infty} f(x) - g(x) = 0.$$

For example, if we consider the hyperbola with equation

$$\frac{(x-a)^2}{A^2} - \frac{(y-b)^2}{B^2} = 1$$

(here $A > 0$ and $B > 0$) and the line with equation

$$\frac{x-a}{A} - \frac{y-b}{B} = 0$$

or the line with equation

$$\frac{x-a}{A} + \frac{y-b}{B} = 0$$

we can show that either line is asymptotic to the hyperbola. One branch of the hyperbola is

$$y = \frac{B}{A}\sqrt{(x-a)^2 - A^2} + b$$

and we will show this is asymptotic to

$$y = \frac{B}{A}(x-a) + b.$$

When $x \geq a + A$ we have

$$\begin{aligned} 0 &\leq \left(\frac{B}{A}(x-a) + b\right) - \left(\frac{B}{A}\sqrt{(x-a)^2 - A^2} + b\right) \\ &= \frac{B}{A}\left((x-a) - \sqrt{(x-a)^2 - A^2}\right) \\ &= \frac{AB}{(x-a) + \sqrt{(x-a)^2 - A^2}} \\ &\leq \frac{AB}{x-a} \end{aligned}$$

so by the Pinching Principle

$$\lim_{x\to\infty} \left(\frac{B}{A}(x-a) + b\right) - \left(\frac{B}{A}\sqrt{(x-a)^2 - A^2} + b\right) = 0$$

showing us the the hyperbola is asymptotic to the line.