

# Properties of limits at infinity

We will now develop some general properties of limits at infinity. These will allow us to determine limits efficiently.

## 1 Some general properties

**Theorem 1 (Shift and Scale)** *Suppose that*

$$\lim_{x \rightarrow \infty} f(x) = L.$$

*Then for any constants  $a$  and  $b$ ,*

$$\lim_{x \rightarrow \infty} af(x) + b = aL + b.$$

*Furthermore*

- *If  $a > 0$  then*

$$\lim_{x \rightarrow \infty} f(ax + b) = L$$

- *If  $a < 0$  then*

$$\lim_{x \rightarrow -\infty} f(ax + b) = L.$$

**Reason:** If  $a = 0$  then  $af(x) + b$  is the constant function  $b$ , which has limit  $b$ , as we demonstrated in our first example.

Suppose that  $a \neq 0$ . Consider  $t > 0$ . We know that there is a constant  $F_{t/|a|}$  so that if  $x > F_{t/|a|}$  then  $|f(x) - L| < t/|a|$ . Therefore, if  $x > F_{t/|a|}$  then

$$|af(x) + b - (aL + b)| = |a||f(x) - L| < |a|\frac{t}{|a|} = t,$$

which is what we needed to know.

Next, put  $g(x) = f(ax + b)$ . In the first case, the domain of  $g$  is still unbounded above, since  $x$  is in the domain of  $g$  if and only if  $ax + b$  is in the domain of  $f$ . Now, consider  $t > 0$ . We know that there is some  $F_t$  so that if  $y > F_t$  then  $|f(y) - L| < t$ . Set  $G_t = (F_t - b)/a$ .  $x > G_t$  if and only if  $ax + b > F_t$  and therefore

$$|g(x) - L| = |f(ax + b) - L| < t$$

as desired.

In the second case, there is a bit of care to be taken owing to  $a$  being negative. The domain of  $g$  is unbounded below since  $x$  is in the domain of  $g$  when  $ax + b$  is in the domain of  $f$ . As before, for each  $t > 0$  there is a number  $F_t$  so that  $y > F_t$  implies  $|f(y) - L| < t$ . Set  $G_t = (F_t - b)/a$ . Then  $x < G_t$  if and only if  $ax + b > F_t$ . Therefore

$$|g(x) - L| = |f(ax + b) - L| < t$$

as desired. **QED**

## 2 More applications

We can now combine the shift and scale principle with the monotone function theorem to compute some limits.

**Application 1:** Show that

$$\lim_{x \rightarrow \infty} \left(\frac{1}{2}\right)^x = 0.$$

**Solution:** Suppose that  $f(x) = (1/2)^x$  for  $x$  any rational number. Observe that  $f(x+1) = f(x)/2$ .  $f(x)$  is decreasing and bounded below by 0. Therefore there is some  $L$  so that

$$\begin{aligned} L &= \lim_{x \rightarrow \infty} f(x) \\ &= \lim_{x \rightarrow \infty} f(x+1) \\ &= \lim_{x \rightarrow \infty} \frac{1}{2} f(x) \\ &= \frac{L}{2} \end{aligned}$$

so  $L = L/2$ . Solving for  $L$  gives  $L = 0$ .

**Application 2:** Suppose that  $r$  is a positive rational number. Show that

$$\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0$$

**Solution:**  $f(x) = x^{-r}$  for  $x > 0$  is decreasing and bounded below by 0, so  $f(x)$  has a limit as  $x$  approaches infinity. Let  $L$  denote this limit.

$$\begin{aligned} L &= \lim_{x \rightarrow \infty} f(x) \\ &= \lim_{x \rightarrow \infty} f(x/2) \\ &= \lim_{x \rightarrow \infty} 2^r f(x) \\ &= 2^r L. \end{aligned}$$

Therefore,  $L = 2^r L$ . Since  $2^r \neq 1$  we must have  $L = 0$ .

**Theorem 2 (Equivalent Conditions)** *Each of the following statements implies the other two.*

$$\lim_{x \rightarrow \infty} f(x) = L \tag{1}$$

$$\lim_{x \rightarrow \infty} f(x) - L = 0 \tag{2}$$

$$\lim_{x \rightarrow \infty} |f(x) - L| = 0. \tag{3}$$

**Reason:** Choose  $t > 0$ .

Suppose that (1) is true. Then (2) follows from Shift and Scale.

Suppose that (2) is true. Then there is a constant  $F_t$  so that if  $x > F_t$  then  $|(f(x) - L) - 0| < t$ . We see that (3) is true, since if  $x > F_t$  then

$$||f(x) - L| - 0| = |f(x) - L| = |(f(x) - L) - 0| < t$$

as required.

Finally, suppose that (3) is true. Then there exists a constant  $F_t$  so that if  $x > F_t$  then  $||f(x) - L| - 0| < t$ . Then we see that (1) is true since

$$|f(x) - L| = ||f(x) - L| - 0| < t$$

as required. **QED**

**Theorem 3 (Maxima)** *If  $f$  and  $g$  have common domain  $D$  and*

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= L \\ \lim_{x \rightarrow \infty} g(x) &= L \end{aligned}$$

then

$$\lim_{x \rightarrow \infty} \max\{f(x), g(x)\} = L.$$

**Reason:** Consider  $t > 0$ . There exists constants  $F_t$  and  $G_t$  so that

- If  $x > F_t$  then  $|f(x) - L| < t$ ;
- If  $x > G_t$  then  $|g(x) - L| < t$ .

Let  $M_t = \max\{F_t, G_t\}$ . Then for  $x > M_t$  we have

$$|\max\{f(x), g(x)\} - L| = |\max\{f(x) - L, g(x) - L\}| \leq \max\{|f(x) - L|, |g(x) - L|\} < t$$

as required. **QED**

This result extends to any finite number of functions by induction.

Next, recall that if  $a \leq b \leq c$  then

$$|b| \leq \max\{|a|, |c|\}.$$

**Theorem 4 (Pinching Principle)** *Suppose that for  $f$ ,  $g$  and  $h$  have common domain  $D$ , an unbounded subset of  $(0, \infty)$ , and  $f(x) \leq g(x) \leq h(x)$  for all  $x \in D$ . If*

$$\lim_{x \rightarrow \infty} f(x) = L = \lim_{x \rightarrow \infty} h(x)$$

then

$$\lim_{x \rightarrow \infty} g(x) = L.$$

**Reason:** Note that since  $f(x) \leq g(x) \leq h(x)$  then  $f(x) - L \leq g(x) - L \leq h(x) - L$ , so

$$|g(x) - L| \leq \max\{|f(x) - L|, |h(x) - L|\}.$$

Pick any  $t > 0$ . As in the last argument, there are constants  $F_t$  and  $H_t$

- If  $x > F_t$  then  $|f(x) - L| < t$ ;
- If  $x > H_t$  then  $|h(x) - L| < t$ .

Put  $G_t = \max\{F_t, H_t\}$ . If  $x > G_t$  then  $x > F_t$  and  $x > H_t$ , and

$$\begin{aligned} |g(x) - L| &\leq \max\{|f(x) - L|, |g(x) - L|\} \\ &\leq t \end{aligned}$$

so this choice of  $G_t$  meets the requirements of the definition of limit. **QED**

## 2.1 An application to sine

For  $x > 2/\pi$  define

$$f(x) = \frac{\sin(1/x)}{1/x}.$$

Then

$$\lim_{x \rightarrow \infty} f(x) = 1.$$

**Reason:** Let  $\theta = 1/x$ , so that  $\theta \in (0, \pi/2)$ . Consider the unit circle  $x^2 + y^2 = 1$ , which contains the point  $(\cos(\theta), \sin(\theta))$ , and the following three geometric figures:

- $T_1$ , the triangular region with these vertices:  $(0, 0)$ ,  $(1, 0)$  and  $(\cos(\theta), \sin(\theta))$ . The area of  $T_1$  is  $\sin(\theta)/2$ .
- $S_\theta$  the sector with central angle  $\theta$ , and these vertices:  $(0, 0)$ ,  $(1, 0)$  and  $(\cos(\theta), \sin(\theta))$ . The area of  $S_\theta$  is  $\theta/2$ , and  $S_\theta$  contains  $T_1$ .
- $T_2$ , the triangular region with these vertices:  $(0, 0)$ ,  $(1, 0)$  and  $(1, \tan(\theta))$ . The area of  $T_2$  is  $\tan(\theta)/2$ , and  $T_2$  contains  $S_\theta$ .

From our geometric construction we know that

$$\frac{\sin(\theta)}{2} \leq \frac{\theta}{2} \leq \frac{\tan(\theta)}{2}$$

so

$$\sin(\theta) \leq \theta \leq \frac{\sin(\theta)}{\cos(\theta)}.$$

From this we conclude that

$$1 \leq \frac{\theta}{\sin(\theta)} \leq \frac{1}{\cos(\theta)} = \frac{1}{(\cos(\theta))^2} = \frac{1}{1 - (\sin(\theta))^2}$$

so

$$1 - (\sin(\theta))^2 \leq \frac{\sin(\theta)}{\theta} \leq 1.$$

Since  $0 < \sin(\theta) < \theta < 1$  we have  $(\sin(\theta))^2 \leq \sin(\theta) < \theta$  so

$$1 - \theta \leq \frac{\sin(\theta)}{\theta} \leq 1.$$

Writing this in terms of  $x$  we have

$$1 - \frac{1}{x} \leq \frac{\sin(1/x)}{1/x} \leq 1.$$

By Shift and Scale the left hand expression has a limit of 1, so by the Pinching Principle, so does the middle expression.

### 3 Sums, Products and Quotients

We assume that  $f$  and  $g$  have a common domain that is unbounded above.

**Theorem 5 (Sums)** *If*

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= F \\ \lim_{x \rightarrow \infty} g(x) &= G \\ \text{then} \\ \lim_{x \rightarrow \infty} f(x) + g(x) &= F + G \end{aligned}$$

**Reason:** Observe that

$$0 \leq |(f(x) + g(x)) - (F + G)| \leq |f(x) - F| + |g(x) - G| \leq 2 \max\{|f(x) - F|, |g(x) - G|\}.$$

The combination of Shift and Scale and Maximum tells us that the rightmost function converges to 0. The Pinching Principle then gives us that the middle function converges to 0. Then the result follows from the Equivalent Conditions. **QED**

Now we turn to products and quotients. We begin with two special cases. The general results follow easily from these.

**Lemma 1** *Suppose that*

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= 0 \\ \lim_{x \rightarrow \infty} g(x) &= 1. \end{aligned}$$

*Then*

$$\begin{aligned} \lim_{x \rightarrow \infty} (f(x))^2 &= 0 \\ \lim_{x \rightarrow \infty} 1/g(x) &= 1. \end{aligned}$$

**Reason:** There is some  $F_1$  so that if  $x > F_1$  then  $|f(x)| < 1$ . Then for  $x > F_1$  we have

$$0 \leq (f(x))^2 < |f(x)|.$$

Since

$$\lim_{x \rightarrow \infty} f(x) = 0$$

we know that

$$\lim_{x \rightarrow \infty} |f(x)| = 0$$

so it follows from the Pinching Principle that

$$\lim_{x \rightarrow \infty} (f(x))^2 = 0.$$

There is some  $G_{1/2}$  so that if  $x > G_{1/2}$  then  $|g(x) - 1| < 1/2$ , which is the same as  $1/2 < g(x) < 3/2$ . Hence if  $x > G_{1/2}$  then

$$0 \leq \left| \frac{1}{g(x)} - 1 \right| = \frac{|g(x) - 1|}{g(x)} \leq 2|g(x) - 1|.$$

Since

$$\lim_{x \rightarrow \infty} g(x) = 1$$

we know that

$$\lim_{x \rightarrow \infty} |g(x) - 1| = 0.$$

Therefore it follows from the Pinching Principle that

$$\lim_{x \rightarrow \infty} \left| \frac{1}{g(x)} - 1 \right| = 0.$$

This is equivalent to

$$\lim_{x \rightarrow \infty} \frac{1}{g(x)} = 1.$$

That wraps up the lemma.

**QED**

**Theorem 6 (Quotients and Products)** *Suppose that*

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= F \\ \lim_{x \rightarrow \infty} g(x) &= G \neq 0 \\ \lim_{x \rightarrow \infty} h(x) &= H. \end{aligned}$$

*Then*

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x)h(x) &= FH \\ \lim_{x \rightarrow \infty} f(x)/g(x) &= F/G \end{aligned}$$

**Reason:** First suppose that  $F = H = 0$ . Observe that

$$f(x)h(x) = \frac{1}{4}(f(x) + h(x))^2 - \frac{1}{4}(f(x) - h(x))^2.$$

(This is called a **polarization identity**.) The right hand side is the sum of two terms, each of which has a limit of 0, so

$$\lim_{x \rightarrow \infty} f(x)h(x) = 0.$$

To get the general case, write

$$f(x)h(x) = (f(x) - F)(h(x) - H) + H(f(x) - F) + F(h(x) - H) + HF.$$

The right hand side has a limit of HF. Thus the statement about products is true.

Now for quotients. We know that

$$\lim_{x \rightarrow \infty} \frac{g(x)}{G} = 1,$$

so

$$\lim_{x \rightarrow \infty} \frac{G}{g(x)} = 1.$$

Therefore, by Shift and Scale,

$$\lim_{x \rightarrow \infty} \frac{1}{g(x)} = \lim_{x \rightarrow \infty} \frac{1}{G} \frac{G}{g(x)} = \frac{1}{G} \lim_{x \rightarrow \infty} \frac{G}{g(x)} = \frac{1}{G}.$$

The general result for quotients now follows from the result for products and the fact that  $f(x)/g(x) = f(x)(1/g(x))$ . **QED**

### 3.1 A method for computing approximations to square roots

Suppose that  $a > 0$  is given and we want to approximate  $\sqrt{a}$ . Here is a recursive method known as Newton's method. We will give a geometric explanation of its origins a little later.

Let  $b > \sqrt{a}$  be given. For example, if  $\sqrt{5}$  is to be estimated, take  $b = 3$ . If  $\sqrt{1/2}$  is to be estimated, take  $b = 1$ . As a general rule, take  $b$  to be one more than the integer part of  $\sqrt{a}$ .

Now, we define an infinite sequence  $r_n$  recursively by

$$\begin{aligned} r_0 &= b \\ r_{n+1} &= \frac{r_n^2 + a}{2r_n} \text{ if } n \geq 0 \end{aligned}$$

For example, if  $a = 1/2$  then

$$\begin{aligned} r_0 &= 1 \\ r_1 &= \frac{1^2 + (1/2)}{2} = \frac{3}{4} = 0.75 \\ r_2 &= \frac{(3/4)^2 + (1/2)}{2 \times (3/4)} = \frac{17}{24} \approx 0.7083333333 \\ r_3 &= \frac{(17/24)^2 + (1/2)}{2 \times (17/24)} = \frac{577}{816} \approx 0.7071078431 \end{aligned}$$

Observe that most calculators give  $\sqrt{1/2} \approx 0.7071067810$ .

Here is why this works. First, we can show by induction that if  $r_n \geq \sqrt{a}$  then  $r_{n+1} \geq \sqrt{a}$ : By assumption,  $r_0 \geq \sqrt{a}$ . Now, suppose that  $r_n \geq \sqrt{a}$ . Then, since  $\sqrt{a} > 0$  we have

$$r_{n+1} - \sqrt{a} = \frac{r_n^2 + a}{2r_n} - \sqrt{a} = \frac{r_n^2 - 2\sqrt{a}r_n + a}{2r_n} = \frac{(r_n - \sqrt{a})^2}{2r_n} \geq 0$$

so  $r_{n+1} \geq \sqrt{a}$ .

Next we show that  $r_n$  is a decreasing function of  $n$ . Note that

$$r_{n+1} - r_n = \frac{r_n^2 + a}{2r_n} - r_n = \frac{a - r_n^2}{2r_n} \leq 0$$

since  $r_n \geq \sqrt{a}$  for any non-negative integer  $n$ . Therefore  $r_n \geq r_{n+1}$  as claimed.

We have now shown that  $r_n$  is decreasing and bounded below, so there is some real number  $L$  so that

$$\lim_{n \rightarrow \infty} r_n = L.$$

It remains to show that  $L = \sqrt{a}$ . Define a new infinite sequence  $s_n$  by

$$s_n = r_{n+1} - \frac{r_n^2 + a}{2r_n}.$$

On the one hand,  $s_n = 0$  for all  $n$ , so

$$\lim_{n \rightarrow \infty} s_n = 0.$$

On the other hand, using our theorems about limits,

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} r_{n+1} - \frac{r_n^2 + a}{2r_n} = L - \frac{L^2 + a}{2L} = \frac{L^2 - a}{2L}$$

so  $L^2 = a$  and  $L \geq 0$ , meaning  $L = \sqrt{a}$ .

## 4 Horizontal Asymptotes of Rational Functions

Suppose we want to evaluate the behavior of a function defined by the ratio of two polynomials. For example, suppose we want to investigate the limiting behavior of

$$f(x) = \frac{x^3 + 6x - 9}{4x^4 + 2x - 11}$$

as  $x$  tends to infinity. Since we are comfortable with the behavior of power functions, we reduce our problem to considering such functions by first factoring out the highest power of  $x$  from the numerator and denominator of  $f(x)$ , assuming that  $x > 0$ , to get

$$f(x) = \frac{x^3(1 + 6x^{-2} - 9x^{-3})}{x^4(4 + 2x^{-3} - 11x^{-4})} = \frac{1}{x} \times \frac{1 + 6x^{-2} - 9x^{-3}}{4 + 2x^{-3} - 11x^{-4}}.$$

We recognize immediately that the limit of the first factor is 0, while, using our algebraic facts about limits of products, sums and quotients, that the limit of the second factor is  $1/4$ , so we know that  $f(x)$  converges 0 as  $x$  approaches infinity.

As a second example, consider the reciprocal of  $f$ :

$$g(x) = \frac{4x^4 + 2x - 11}{x^3 + 6x - 9}.$$

We employ the same strategy. This time we get for  $x > 0$ :

$$g(x) = f(x) = \frac{x^4(4 + 2x^{-3} - 11x^{-4})}{(x^3)(1 + 6x^{-2} - 9x^{-3})} = x \frac{4 + 2x^{-3} - 11x^{-4}}{1 + 6x^{-2} - 9x^{-3}}.$$

Since the second factor has a limit of 4 while the first factor diverges to infinity, we see that  $g(x)$  diverges to infinity as  $x$  approaches infinity.

Note that this same strategy applies equally well if we have fractional powers.