Limits of monotone functions

We will now show that bounded monotone functions have limits. As an application we will revisit compound interest.

1 The basic theorem

Theorem 1 (Monotone Functions) *Suppose that f is a function with range R and its domain is unbounded above.*

If f is non-decreasing and bounded above then

$$\lim_{x\to\infty} f(x) = \mathrm{lub}(R).$$

If f is non-increasing and bounded below then

$$\lim_{x\to\infty} f(x) = \mathrm{glb}(R).$$

Reason: Here is the argument for the former case. Let $L = \mathrm{lub}(R)$. and let $t > 0$. Then there is some x_t so that

$$|L - f(x_t)| = L - f(x_t) < t,$$

for otherwise

$$f(x) \leq L - t < L$$

for every x in the domain of f, $L - t$ is a smaller upper bound for the range of f than L is. Put $F_t = x_t$. Since f is non-decreasing, if $x > F_t$ then

$$|f(x) - L| = L - f(x) \leq L - f(x_t) < t$$

as desired.

The argument for the latter case is similar. **QED**

2 The Natural Exponential Function for non-negative real numbers

For a fixed real number r the infinite sequence for $n > 0$ given by

$$E_n(r) := \left(1 + \frac{r}{n}\right)^n$$

for all positive integers n arises in the study of compound interest in algebra courses, where r represents the interest rate in decimal form. For example

$$E_n(0.05) = \left(1 + \frac{0.05}{n}\right)^n$$

is the return on one dollar invested for n years at 5% per year compounded annually.

It is asserted (without proof)

$$\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n = \mathrm{e}^r$$

where

$$\mathrm{e} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

As we work through the material on limits at infinity we will supply the details to clarify and support these assertions.

We begin by showing that $E_n(r)$ has a limit as $n \to \infty$ for any non-negative real number r. We denote this limit by $\exp(r)$. The function $r \to \exp(r)$ is called the natural exponential function.

Since $E_n(0) = 1$ for each n, we have $\exp(0) = 1$.

Applying the binomial formula to $E_n(r)$ we see that for $n \geq 2$ that we have

$$E_n(r) \quad = \quad 1 + r + \sum_{k=2}^{n} \frac{r^k}{k!} \prod_{j=1}^{k-1} \left(1 - \frac{j}{n}\right). \tag{1}$$

If $r > 0$ then (1) shows that the sequence $E_n(r)$ is strictly increasing since for each positive integer j, the ratio j/n is decreasing as n increases. Furthermore, since

$$1 - \frac{j}{n} \leq 1$$

if $j \in [0, n]$ we see that

$$E_n(r) \quad \leq \quad 1 + \sum_{k=1}^{n} r^k. \tag{2}$$

Therefore, if $0 \leq r < 1$ we have

$$1 + r \leq E_n(r) \leq \frac{1}{1-r}. \tag{3}$$

Furthermore, since for $r \geq 0$ and $s \geq 0$ we have $E_n(r+s) \leq E_n(r)E_n(s)$, it follows from (3) that for each positive integer N and each $0 \leq r < 1$

$$E_n(Nr) \leq \left(\frac{1}{1-r}\right)^N.$$

Hence $E_n(r)$ is a bounded sequence for any $r \geq 0$. Thus we have

Proposition 1 *The sequence $E_n(r)$ is convergent for each $r \geq 0$.*