Limits at infinity

1 The definition of limit at infinity

We will first consider **limits at infinity**, which should seem familiar from the study of horizontal asymptotes in algebra and from the relation between compound interest and the exponential function. We will revisit each of these topics.

Suppose that the domain of a function f is unbounded above and L is a real number. We say that the **limit** of $f(x)$ as x **approaches infinity** is L, written

$$\lim_{x\to\infty} f(x) = L$$

if for every $t > 0$ there is a real number F_t so that if x is in the domain of f and $x > F_t$ then

$$|f(x) - L| < t.$$

If there is no limit of $f(x)$ as x approaches infinity, we say that $f(x)$ fails to converge. Sometimes we can be more specific. If for every real number M there is a real number F_M so that for $x > F_M$ we have have $f(x) > M$ then we say that that $f(x)$ **diverges to positive infinity** and we write

$$\lim_{x\to\infty} f(x) = \infty.$$

For example x^2 diverges to infinity as x approaches infinity. Take $F_M = 1 + |M|$.

Another similar possibility is that for every real number M there is real number F_M so that if $x > F_M$ then $f(x) < M$. In this case we say that $f(x)$ **diverges to negative infinity** and we write

$$\lim_{x\to\infty} f(x) = -\infty.$$

As you might guess, $-x^3$ diverges to negative infinity as x approaches infinity. Take $F_M = 1 + |M|$.

We cannot say often enough that the symbols ∞ and $-\infty$ do not represent numbers.

Important: For a given function f there cannot be two different numbers satisfying the limit definition. For suppose $L_1 < L_2$ did so. Put $t = (L_2 - L_1)/3$. According to the limit definition there must be a number F_1 so that $x > F_1$ implies $f(x) \in (L_1 - t, L_1 + t)$ and a number F_2 so that $x > F_2$ implies $f(x) \in (L_2 - t, L_2 + t)$. This is impossible since

$$(L_1 - t, L_1 + t) \cap (L_2 - t, L_2 + t) = \emptyset.$$

Along the same lines is the observation that if $f(x) \le A$ for all $x > B$ and in the domain of f, and

$$\lim_{x\to\infty} f(x) = L$$

then $L \le A$. To see why, suppose that $L > A$. Let $t = (L - A)/2 > 0$. Then there is some F_t such that if $x > F_t$ then $|f(x) - L| > t$. This means that if $x > \max\{B, F_t\}$ then

$$f(x) > L - t = L - \frac{L-A}{2} = \frac{L+A}{2} > \frac{A+A}{2} = A.$$

This contradicts our assumptions that if $x > B$ then $f(x) \le A$.

2 Some examples

In the examples below we assume that the domain of f is an unbounded subset of the positive real numbers unless we state otherwise.

Constant functions: Suppose that a is a real number and $f(x) = a$. Then

$$\lim_{x\to\infty} f(x) = a.$$

Here we can take $F_t = 0$ for any $t > 0$ since

$$|f(x) - a| = |a - a| = 0 < t.$$

This result is true for any function constant on an unbounded subset of $(0, \infty)$.

A function with no limit: Suppose that $f(x) = (-1)^x$ where x is any positive integer. Then there is no limit for $f(x)$ as x approaches infinity. Simply consider $t = 1/2$.

The reciprocal function: Suppose that $f(x) = 1/x$, $x > 0$. Then

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

This is easy to see. Given $t > 0$, if $x > 1/t$, then

$$|f(x) - 0| = \frac{1}{x} < \frac{1}{1/t} = t$$

as required.

3 Limits at minus infinity

If the domain of f is unbounded below we say that **the limit of $f(x)$ as x approaches minus infinity is L**, written

$$\lim_{x\to-\infty} f(x) = L$$

if for every $t > 0$ there is a number F_t such that if $x < F_t$ then

$$|f(x) - L| < t.$$

It is easy to see that

$$\lim_{x\to-\infty} f(x) = L$$

if and only if

$$\lim_{x\to\infty} f(-x) = L$$

since $x < F$ if and only if $-x > -F$.