

# Limits

## 1 Functions, bounded and unbounded sets

Before we begin our discussion of functions it will be helpful to have defined some terminology and notation for sets and functions. Unless there are statements to the contrary, all sets are assumed to be subsets of the real numbers.

### 1.1 Bounded and unbounded sets

We say that a subset  $D$  of the real numbers is

- **unbounded above** if given any real number  $r$  we can find  $d \in D$  so that  $d > r$ ;
- **bounded above** if there is a real number  $U$  so that  $d \leq U$  for every  $d \in D$ , and  $U$  is said to be an **upper bound** for  $D$ ;
- **unbounded below** if given any real number  $r$  we can find  $d \in D$  so that  $d < r$ ;
- **bounded below** if there is a real number  $L$  so that  $d \geq L$  for every  $d \in D$  and  $L$  is said to be a lower bound for  $D$ .
- **bounded** if it is bounded above and below.

For example, the positive integers are bounded below and unbounded above. The negative real numbers are bounded above and unbounded below.

If  $D$  is bounded above, we say that the real number  $U$  is the **least upper bound** of  $D$  if both of the following are true:

- $U$  is an upper bound of  $D$ ;
- if  $U'$  is any other upper bound of  $D$  then  $U \leq U'$ .

Similarly, if  $D$  is bounded below, we say that the real number  $L$  is the **greatest lower bound** of  $D$  if both of the following are true:

- $L$  is a lower bound of  $D$ ;
- if  $L'$  is any other lower bound of  $D$  then  $L \geq L'$ .

A set can have at most one least upper bound and at most one greatest lower bound. The least upper bound and greatest lower bound of any bounded interval are its endpoints.

Some notation: The least upper bound of  $D$  is denoted by  $\text{lub}(D)$  and the greatest lower bound is denoted by  $\text{glb}(D)$ .

The important fact, which we cannot take the time to prove in this course, is

**Theorem 1** *If  $D$  is bounded above then it has a least upper bound, and if  $D$  is bounded below then it has a greatest lower bound.*

A proof of this theorem relies on the construction of the real numbers, which we shall not attempt here. See **Principles of Mathematical Analysis** by Walter Rudin.

## 2 Functions

Suppose that  $A$  and  $B$  are any non empty sets.  $A \times B$  stands for the set of all ordered pairs  $\{(a, b) : a \in A, b \in B\}$ . For example, if  $A = \{1, 2\}$  and  $B = \{3, 4, 5\}$  then

$$A \times B = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)\}.$$

If  $A$  and  $B$  are the real numbers then  $A \times B$  is what we think of as the plane.

Given  $A$  and  $B$ , a **relation** on  $A \times B$  is just a subset of  $A \times B$ . For example, if  $A$  and  $B$  are the real numbers,

$$C = \{(x, y) : x^2 + y^2 = 1, x \in A, y \in B\}$$

is a relation that we recognize as the unit circle from trigonometry.

Given  $A$  and  $B$ , a function  $f$  from  $A$  to  $B$ , denoted by  $f : A \rightarrow B$  is a relation  $f \subset A \times B$  with the following two properties:

- For each  $a \in A$  there is some  $b \in B$  so that  $(a, b) \in f$ ;
- if  $(a, b) \in f$ ,  $(c, d) \in f$  and  $a = c$  then  $b = d$ .

$A$  is called the **domain** of  $f$  and  $B$  is called the **target** or **co-domain** of  $f$ . If  $(a, b) \in f$  we say that  $b$  is the **value** of  $f$  at  $a$  and we write  $b = f(a)$ . The set of all values of  $f$  is called the **range** of  $f$ . If the range of  $f$  is bounded above, then we say  $f$  is **bounded above**. Similarly for bounded below and bounded.

If the range of  $f$  is contained in the real numbers, we say that  $f$  is **real valued**. If the domain of  $f$  is an unbounded subset of the non-negative integers we say that  $f$  is an **infinite sequence**. The most common domains for infinite sequences are the positive integers and the non-negative integers, but it is possible to have infinite sequences with other domains, such as the prime numbers.

Finally, if the domain and co-domain of  $f$  are subsets of the real numbers, we say that  $f$  is:

- **increasing** if for each pair  $a < b$  implies  $f(a) \leq f(b)$ ;
- **strictly increasing** if for each pair  $a < b$  implies  $f(a) < f(b)$ ;
- **decreasing** if for each pair  $a < b$  implies  $f(a) \geq f(b)$ ;
- **strictly decreasing** if for each pair  $a < b$  implies  $f(a) > f(b)$ .
- **monotone** if one of the preceding four properties hold.