# Inverse Trigonometric Functions

We are now in a position to show that the inverse trigonometric functions are differentiable. We can reduce this to demonstration to showing that the arctangent function is differentiable owing to the following identities:

$$\begin{aligned}
\operatorname{arccot}(x) &= \frac{\pi}{2} - \arctan(x) \\
\arcsin(x) &= \arctan\left(\frac{x}{\sqrt{1-x^2}}\right) \\
\arccos(x) &= \frac{\pi}{2} - \arcsin(x) \\
\operatorname{arcsec}(x) &= \arccos\left(\frac{1}{x}\right) \\
\operatorname{arccsc}(x) &= \frac{\pi}{2} - \operatorname{arcsec}(x)
\end{aligned}$$

We have first the following theorem.

**Theorem 1** *Suppose that $g$ is the inverse of $f$, that $f$ is differentiable at $x = a$ with $f'(a) \neq 0$ and $g$ is continuous at $f(a)$. Then $g$ is differentiable at $f(a)$ and $g'(f(a))f'(a) = 0$. Furthermore, if $f'(a) = 0$ then $g$ is not differentiable at $f(a)$.*

**Reason:** The logic here is similar to that in the proof of the chain rule. Let $b = f(a)$, so $g(b) = a$. Then for $y \neq b$ and $y$ in the domain of $g$,

$$1 = \frac{y-b}{y-b} = \frac{f(g(y)) - f(g(b))}{y-b} = \frac{f(g(y)) - f(g(b))}{g(y) - g(b)} \times \frac{g(y) - g(b)}{y-b}.$$

In other words,

$$\frac{g(y) - g(b)}{y-b} = \left(\frac{f(g(y)) - f(a))}{g(y) - a)}\right)^{-1}. \tag{1}$$

Since $g$ is continuous at $b$ and $f$ is differentiable at $a$ with $f'(a) \neq 0$, the left right hand side of (1) converges to $1/f'(a)$ as $y$ approaches $a$, telling us that

$$\lim_{y \to f(a)} \frac{g(y) - g(f(a))}{y - f(a)} = \frac{1}{f'(a)}$$

as claimed. **QED**

We can easily show that this theorem applies to the arctangent function by an application of the mean value theorem, which tells us that for any $-\pi/2 < a < b < \pi/2$ there is some $a < c < b$ such that

$$\tan(b) - \tan(a) = \sec^2(c)(b-a).$$

First, since $\sec^2(c) \geq 1$ regardless of the value of $c$ we have

$$|\tan(b) - \tan(a)| = \sec^2(c)|b-a| \geq |b-a|.$$

Now, putting $a = \arctan(x)$ and $b = \arctan(y)$ we have

$$|\arctan(y) - \arctan(x)| \leq |y-x|.$$

By the Pinching Principle we see that arctangent is continuous (in fact, uniformly continuous) on its entire domain, and, by the theorem, differentiable. To compute the derivative, according to the theorem we have

$$\frac{d}{dy}\arctan(y) = \frac{1}{\sec^2(\arctan(y))} = \frac{1}{1+y^2}$$

since $1 + \tan^2(u) = \sec^2(u)$.

Alternatively, since $\tan(\arctan(x)) = x$, applying the chain rule we have

$$\begin{aligned}
1 &= \sec^2(\arctan(x))\frac{d}{dx}\arctan(x) \\
1 &= (1+\tan^2(\arctan(x)))\frac{d}{dx}\arctan(x) \\
\frac{1}{1+x^2} &= \frac{d}{dx}\arctan(x).
\end{aligned}$$

Immediately we see that

$$-\frac{1}{1+x^2} = \frac{d}{dx}\text{arccot}(x).$$

Now, since the differentiability of arctan implies that of arcsin, the chain rule, along with the fact that the range of arcsin is $[-\pi/2, \pi/2]$, shows that for $|x| < 1$

$$\begin{aligned}
x &= \sin(\arcsin(x)) \\
1 &= \cos(\arcsin(x))\frac{d}{dx}\arcsin(x) \\
1 &= \sqrt{1-\sin^2(\arcsin(x))}\frac{d}{dx}\arcsin(x) \\
\frac{1}{\sqrt{1-x^2}} &= \frac{d}{dx}\arcsin(x)
\end{aligned}$$

Immediately we have

$$-\frac{1}{\sqrt{1-x^2}} = \frac{d}{dx}\arccos(x).$$

We turn finally to arcsec and arccsc. Suppose that $|x| > 1$.

$$\begin{aligned}
\frac{d}{dx}\text{arcsec}(x) &= \frac{d}{dx}\arccos(x^{-1}) \\
&= -\frac{1}{1-(1/x)^2}(-x^{-2}) \\
&= \frac{1}{x^2\sqrt{1-(1/x)^2}} \\
&= \frac{1}{|x|\sqrt{x^2-1}}.
\end{aligned}$$

Here we use that $x^2 = |x|^2$ and $|x| = \sqrt{x^2}$. So, finally we have

$$\frac{d}{dx}\text{arccsc}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

for $|x| > 1$.

In summary,

$$\begin{aligned}
\frac{d}{dx}\arctan(x) &= \frac{1}{1+x^2} \\
\frac{d}{dx}\text{arccot}(x) &= -\frac{1}{1+x^2} \\
\frac{d}{dx}\arcsin(x) &= \frac{1}{\sqrt{1-x^2}} \quad |x| < 1 \\
\frac{d}{dx}\arccos(x) &= -\frac{1}{\sqrt{1-x^2}} \quad |x| < 1 \\
\frac{d}{dx}\text{arcsec}(x) &= \frac{1}{|x|\sqrt{x^2-1}} \quad |x| > 1 \\
\frac{d}{dx}\text{arccsc}(x) &= -\frac{1}{|x|\sqrt{x^2-1}} \quad |x| > 1
\end{aligned}$$

For practical purposes, the important formulae are those for arctangent, arcsine and arcsecant.