Integration

1 The Definition the Definite Integral

In order to make precise the preceding discussion will take some doing. Suppose that f is a function that is bounded on an closed interval $[a,b]$, and that N is a positive integer. Since f is bounded, we know that there are constants M and m so that $m \le f(x) \le M$ for any $x \in [a,b]$.

A **partition**, P, of $[a,b]$ is a finite non-decreasing sequence p_k, $k \in \{0,1,\ldots,N\}$ with $p_0 = a$ and $p_N = b$. Given a partition P, for each $n \in \{1,\ldots,N\}$ define

$$\begin{aligned} m_n &= \text{glb}\{f(x) : p_{n-1} \le x \le p_n\} \le M \\ M_n &= \text{lub}\{f(x) : p_{n-1} \le x \le p_n\} \ge m. \end{aligned}$$

For example, if $[a,b] = [0,1]$ and $f(x)$ is non-decreasing, we might take $p_n = n/N$ and then we would have $m_n = f(n-1/N)$ and $M_n = f(n/N)$.

Next, given a partition P define two sums:

$$\begin{aligned} L(f,P) &= m_1(p_1 - p_0) + m_2(p_2 - p_1) + \cdots + m_N(p_N - p_{N-1}) \\ &= \sum_{n=1}^{N} m_n\,(p_n - p_{n-1}) \\ &\le M(b-a) \\ U(f,P) &= M_1(p_1 - p_0) + M_2(p_2 - p_1) + \cdots + M_N(p_N - p_{N-1}) \\ &= \sum_{n=1}^{N} M_n\,(p_{n+1} - p_n) \\ &\ge m(b-a) \end{aligned}$$

It seems reasonable that our area approximations should improve if we add a point to the partition. To make this observation concrete, we introduce a new concept. Recalling that a partition is a function, we say that the partition Q is a refinement of the partition P if the range of P is contained in the range of Q. For example, if the interval is $[0,1]$ and P is the partition $k/4$, $k \in \{0,1,2,3,4\}$, that is

$$0 < \frac{1}{4} < \frac{1}{2} < \frac{3}{4} < 1,$$

then the partition Q given by $k/8$, $k \in \{0,1,2,\ldots,8\}$, that is,

$$0 < \frac{1}{8} < \frac{1}{4} < \frac{3}{8} < \frac{1}{2} < \frac{5}{8} < \frac{3}{4} < \frac{7}{8} < 1,$$

is a refinement of P, but the partition given by $k/6$, $k \in \{0,1,\ldots,6\}$, that is

$$0 < \frac{1}{6} < \frac{1}{3} < \frac{1}{2} < \frac{2}{3} < \frac{5}{6} < 1,$$

is not a refinement of P. The important fact about refinements is

Proposition 1 *If Q is a refinement of P then*

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P).$$

Furthermore, if P and R are any partitions of $[a,b]$ then

$$L(f,P) \le U(f,R).$$

Reason: The second assertion follows from the first by observing that if P and R are any partitions of $[a,b]$ then we can construct a partition Q whose range is the union of the range of P and the range of R. Then Q is a refinement both of P and of R.

To prove the first assertion observe that is sufficient to consider the case where the range of Q has one more element than the range of P. For this to be true we must have $a < b$.

Suppose the points in P are given by the sequence p_k, $k \in \{0, 1, \ldots, N\}$ and the points in Q are given by the sequence q_k, $k \in \{0, 1, \ldots, N+1\}$. Let q be the element of the range of Q that is not in the range of P. We know that $q \in (a, b)$, so for some positive integers n and m we have

$$p_{n-1} = q_{m-1} < q = q_m < q_{m+1} = p_n.$$

In other words,

$$[q_{m-1}, q] \cup [q, q_{m+1}] \quad = \quad [p_{n-1}, p_n]$$

Observe that

$$\begin{aligned} m_n = \text{glb}\{f(x) : p_{n-1} \leq x \leq p_n\} &\leq \text{glb}\{f(x) : q_{m-1} \leq x \leq q\} := m'; \\ \text{glb}\{f(x) : p_{n-1} \leq x \leq p_n\} &\leq \text{glb}\{f(x) : q \leq x \leq q_{m+1}\} := m''; \\ M_n = \text{lub}\{f(x) : p_{n-1} \leq x \leq p_n\} &\geq \text{lub}\{f(x) : q_{m-1} \leq x \leq q\} := M' \\ \text{lub}\{f(x) : p_{n-1} \leq x \leq p_n\} &\geq \text{lub}\{f(x) : q \leq x \leq q_{m+1}\} = M'' \end{aligned}$$

Therefore

$$\begin{aligned} L(f, Q) - L(f, P) &= m'(q - q_{m-1}) + m''(q_{m+1} - q) - m_n(p_n - p_{n-1}) \\ &= (m' - m_n)(q - q_{m-1}) + (m'' - m_n)(q_{m+1} - q) \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned} U(f, Q) - U(f, P) &= M'(q - q_{m-1}) + M''(q_{m+1} - q) - M_n(p_n - p_{n-1}) \\ &= (M' - M_n)(q - q_{m-1}) + (M'' - M_n)(q_{m+1} - q) \\ &\leq 0 \end{aligned}$$

This what we needed to show. **QED**

Put

$$\begin{aligned} \underline{\int_a^b} f(x)\, dx &:= \text{lub}\{L(f, P), P \text{ a partition of } [a, b]\} \\ \overline{\int_a^b} f(x)\, dx &:= \text{glb}\{U(f, R), R \text{ a partition of } [a, b]\} \end{aligned}$$

These quantities are called the **lower integral** and **upper integral** of f on $[a, b]$.

It follows from Proposition 1 that for any partition R, $U(f, R)$ is an upper bound for $\{L(f, P),\ P$ a partition of $[a, b]\}$ and for any partition P $L(f, P)$ is a lower bound for $\{U(f, R),\ R$ a partition of $[a, b]\}$, so the lower integral of f on $[a, b]$ cannot be larger than the upper integral of f on $[a, b]$. If the upper and lower integrals coincide, then the common value is called the **integral** of f on $[a, b]$ and is denoted by

$$\int_a^b f(x)\, dx.$$

The symbol x can be replaced by any other symbol, so we may write

$$\int_a^b f(y)\, dy$$

for example. Occasionally you may see

$$\int_{[a,b]} f(x)\, dx.$$

2 The change of variables formula

It is clear that some integrals are easier to evaluate than others. The change of variables formula tells us when two integrals are equal. It is the analog of the chain rule. We will give a formulation that is not quite the most general.

Theorem 1 (Change of Variables Theorem) *Suppose that u' is continuous on $[a, b]$, that u is non-decreasing, and that f is continuous on $[u(a), u(b)]$. Then*

$$\int_{u(a)}^{u(b)} f(y)\, dy = \int_a^b f(u(x))u'(x)\, dx.$$

Why is this true? Let P be a partition of $[a, b]$. This means we have

$$a = p_0 \leq p_1 \leq \cdots \leq p_{N-1} \leq p_N = b.$$

Put $q_k = u(p_k)$. Then the sequence Q given by $(q_0, q_1, \ldots, q_n)$ is a partition of $[u(a), u(b)]$. Since f is continuous on $[u(a), u(b)]$ it is integrable there, and $f(u(x))u'(x)$ is continuous on $[a, b]$ so it is integrable there. Applying the mean value theorem to u on $[p_k, p_{k+1}]$ says that there is some $c_k \in (p_k, p_{k+1})$ so that

$$q_{k+1} - q_k = u(p_{k+1}) - u(p_k) = u'(c_k)(p_{k+1} - p_k)$$

Put $y_k = u(c_k)$ to get

$$f(y_k)\,(q_{k+1} - q_k) = f(u(c_k))u'(c_k)(p_{k+1} - p_k)$$

Summing on k we have

$$\sum_{k=0}^{N-1} f(y_k)\,(q_{k+1} - q_k) = \sum_{k=0}^{N-1} f(u(c_k))u'(c_k)(p_{k+1} - p_k).$$

Letting $N \to \infty$ on each side gives

$$\int_{u(a)}^{u(b)} f(y)\, dy = \int_a^b f(u(x))u'(x)\, dx$$

as desired. **QED**