Area and anti-derivatives

Suppose that f is a positive, increasing function on the closed interval $[a, b]$, and that we want to find the area enclosed by the following four curves:

- The line segment from $(a, 0)$ to $(b, 0)$.
- The line segment from $(a, 0)$ to $(a, f(a))$.
- The line segment from $(b, 0)$ to $(b, f(b))$.
- The graph of $y = f(x)$ for $a \leq x \leq b$.

Suppose in addition that $F(x)$ is an anti-derivative of f and that F is continuous on $[a, b]$. For example, we might have $[a, b] = [1, 2]$, $f(x) = 2x \exp(x^2)$ and $F(x) = \exp(x^2)$.

We could estimate this area in the following manner. Pick numbers $p_0 \leq p_1 \leq \ldots \leq p_n$ with $p_0 = a$ and $p_n = b$. Such a finite increasing sequence is called a **partition** of $[a, b]$. From each point $(p_k, 0)$ draw a vertical line up to $y = f(x)$. This line intersects $y = f(x)$ at $(p_k, f(p_k))$. Now we have divided up the region in question into strips. Label these as S_1, S_2, up to S_n where the vertices of S_k, listed in counter-clockwise order are

- $(p_k, 0)$;
- $(p_k, f(p_k))$;
- $(p_{k-1}, f(p_{k-1}))$;
- $(p_{k-1}, 0)$.

For example, the vertices of S_1 are $(p_1, 0)$, $(p_1, f(p_1))$, $(p_0, f(p_0))$ and $(p_0, 0)$.

Since f is increasing it is easy to inscribe and circumscribe each of these strips with rectangles. For example, S_0 contains the rectangle with vertices

- $(p_1, 0)$;
- $(p_1, f(p_0))$;
- $(p_0, f(p_0))$;
- $(p_{k-1}, 0)$;

and is itself is contained in the rectangle with vertices

- $(p_1, 0)$;
- $(p_1, f(p_1))$;
- $(p_0, f(p_1))$;
- $(p_0, 0)$.

This means that the area of this rectangle lies between

$$L_1 := f(p_0)(p_1 - p_0)$$

and

$$U_1 := f(p_1)(p_1 - p_0)$$

More generally, the strip S_k contains the rectangle with vertices

- $(p_k, 0)$;
- $(p_k, f(p_{k-1}))$;
- $(p_{k-1}, f(p_{k-1}))$;
- $(p_{k-1}, 0)$.

and is contained in the rectangle whose vertices are

- $(p_k, 0)$;
- $(p_k, f(p_k))$;
- $(p_{k-1}, f(p_k))$;
- $(p_{k-1}, 0)$,

so its area lies between

$$\begin{aligned} L_k &:= f(p_{k-1})(p_k - p_{k-1}) \\ \text{and} & \\ U_k &:= f(p_k)(p_k - p_{k-1}). \end{aligned}$$

If we let A denote the area in question, we see that

$$L_1 + L_2 + \cdots + L_n \leq A \leq U_1 + U_2 + \cdots + U_n.$$

So far, we have not used that f has a continuous anti-derivative F. However, according to the mean value theorem, for each k there is some q_k so that $p_{k-1} < q_k < p_k$ and

$$F(p_k) - F(p_{k-1}) = f(q_k)(p_k - p_{k-1}).$$

This means that

$$f(p_{k-1})(p_k - p_{k-1}) \leq f(q_k)(p_k - p_{k-1}) \leq f(p_k)(p_k - p_{k-1}).$$

Therefore

$$\begin{aligned} & L_p + L_{p-1} + \cdots + L_2 + L_1 \\ \leq\; & F(b) - F(p_{n-1}) + F(p_{n-1}) - F(p_{n-2}) + \cdots + F(p_2) - F(p_1) + F(p_1) - F(a) \\ \leq\; & U_p + U_{p-1} + \cdots + U_2 + U_1, \end{aligned}$$

or, in simplified form,

$$\begin{aligned} L_p + L_{p-1} + \cdots + L_2 + L_1 &\leq F(b) - F(a) \\ &\leq U_p + U_{p-1} + \cdots + U_2 + U_1. \end{aligned}$$

This suggests that $A = F(b) - F(a)$. We can show this by using the Pinching Principle. Suppose we choose

$$p_k = a + \frac{b-a}{n} k.$$

This is called the **uniform partition** of $[a, b]$ because

$$p_k - p_{k-1} = \frac{b-a}{n},$$

meaning each strip has the same width. Observe then that

$$U_k - L_k = (f(p_k) - f(p_{k-1})) \frac{b-a}{n},$$

so

$$\begin{aligned} & U_n + U_{n-1} + \cdots + U_2 + U_1 - (L_n + L_{n-1} + \cdots + L_2 + L_1 \\ =\; & U_n - L_n + U_{n-1} - L_{n-1} + \cdots + U_2 - L_2 + U_1 - L_1 \\ =\; & \frac{b-a}{n} (f(b) - f(p_{k-1}) + f(p_{k-1}) - f(p_{k-2}) + \cdots + f(p_2) - f(p_1) + f(p_1) - f(a)) \\ =\; & \frac{b-a}{n} (f(b) - f(a)). \end{aligned}$$

Therefore

$$\begin{aligned} 0 &\leq |A - (F(b) - F(a))| \\ &\leq U_n + U_{n-1} + \cdots + U_2 + U_1 - (L_n + L_{n-1} + \cdots + L_2 + L_1 \\ &= \frac{b-a}{n}(f(b) - f(a)). \end{aligned}$$

It then follows from the Pinching Principle that by letting n tend to infinity that

$$A = F(b) - F(a).$$

Returning to our example, we see that the area, A, of the region bounded by

- $y = 2x\exp(x^2)$;
- $y = 0$;
- $x = 1$;
- $x = 2$

is given by

$$A = \exp(4) - \exp(1).$$

The analysis presented above applies equally well if $f(x)$ is decreasing. An important example is when $f(x) = 1/x$, $a = 1$ and $b > 1$. For then, since an anti-derivative of $1/x$ is $\ln(|x|)$, we have a way to estimate logarithms. For example, suppose we want to compute $\ln(2)$. We consider the area of the region bounded by

- $y = 1/x$;
- $y = 0$;
- $x = 1$;
- $x = 2$.

The area of this region, A, is given by

$$A = \ln(2) - \ln(1) = \ln(2).$$

On the other hand, if we take $n = 10$, for example, and use the uniform partition of $[1, 2]$:

$$1 < \frac{11}{10} < \frac{12}{10} < \frac{13}{10} < \frac{14}{10} < \frac{15}{10} < \frac{16}{10} < \frac{17}{10} < \frac{18}{10} < \frac{19}{10} < 2$$

we see that

$$\begin{aligned} & \frac{10}{11}\frac{1}{10} + \frac{10}{12}\frac{1}{10} + \frac{10}{13}\frac{1}{10} + \frac{10}{14}\frac{1}{10} + \frac{10}{15}\frac{1}{10} + \frac{10}{16}\frac{1}{10} + \frac{10}{17}\frac{1}{10} + \frac{10}{18}\frac{1}{10} + \frac{10}{19}\frac{1}{10} + \frac{10}{20}\frac{1}{10} \\ < & \ln(2) \\ < & \frac{1}{1}\frac{1}{10} + \frac{10}{11}\frac{1}{10} + \frac{10}{12}\frac{1}{10} + \frac{10}{13}\frac{1}{10} + \frac{10}{14}\frac{1}{10} + \frac{10}{15}\frac{1}{10} + \frac{10}{16}\frac{1}{10} + \frac{10}{17}\frac{1}{10} + \frac{10}{18}\frac{1}{10} + \frac{10}{19}\frac{1}{10}, \end{aligned}$$

which simplifies to

$$\begin{aligned} & \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20} \\ < & \ln(2) \\ < & \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19}. \end{aligned}$$

These upper and lower estimates of $\ln(2)$ differ by $1/20$. In terms of decimal approximations we have

$$0.668 < \ln(2) < 0.719.$$

If we change from $n = 10$ to $n = 100$ we get (in terms of decimals)

$$0.6906 < \ln(2) < 0.6957.$$

and the estimates differ by $1/200$.