Comparing rates of growth of logatithmic and exponential functions

1 Relative rate of growth

We will say that $f(x)$ grows faster than $g(x)$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

In what follows we will show that exponential functions grow faster than power functions and power functions grow faster than logarithmic functions.

We only need to consider the natural logarithm and exponential functions since for any $a > 0$,

$$\begin{aligned} a^x &:= \exp(\ln(a)x) \\ \log_a(x) &= \frac{\ln(x)}{\ln(a)}. \end{aligned}$$

Suppose that p and q are positive rational numbers. We wish to investigate the limits of the following expressions:

- As x tends to infinity:
$$\frac{x^p}{\exp(x^q)};$$
- As x tends to infinity:
$$\frac{\ln(x)}{x^p};$$
- As x tends to 0 from above:
$$x^p \ln(x) = \frac{\ln(x)}{1/x^p}$$

In each case we will show that the limit is 0. The conclusion is that exponential functions grow most rapidly, followed by power functions and then by logarithmic functions as x tends to infinity, while as x tends to 0, $1/x^p$ grows more rapidly than $-\ln(x)$.

1.1 $x^p/\exp(x^q)$

Observe that if $x > 0$ then

$$0 < \frac{x^p}{\exp(x^q)}$$

and, futhermore, for any positive integer, n, and any $r > 0$,

$$\left(1 + \frac{r}{n}\right)^n < \exp(r).$$

Choose n so that $nq > p$. Then for every $x > 0$:

$$0 < \frac{x^p}{\exp(x^q)} < \frac{x^p}{\left(1 + \frac{x^q}{n}\right)^n} = \frac{x^p}{x^{nq}} \frac{1}{\left(\frac{1}{x^q} + \frac{1}{n}\right)^n} = \frac{1}{x^{nq-p}} \frac{1}{\left(\frac{1}{x^q} + \frac{1}{n}\right)^n}.$$

In summary:

$$0 < \frac{x^p}{\exp(x^q)} < \frac{1}{x^{nq-p}} \frac{1}{\left(\frac{1}{x^q} + \frac{1}{n}\right)^n}. \quad (1)$$

Since the extreme expressions in (1) converge to 0 as x tends to infinity, the Pinching Principle allows us to conclude that p and q are positive rational numbers then

$$\lim_{x\to\infty} \frac{x^p}{\exp(x^q)} = 0$$

as well. For example,

$$\lim_{x\to\infty} (x^2 + 3x - 9)\exp(-x^2) = 0.$$

1.2 $x^p \ln(x)$

Here we want to let x approach 0 from above. Put $f(x) = x^p \ln(x)$. We know that for $0 < x < 1$ that $f(x) < 0$. We will show that $f(x)$ increases as x approaches 0 from above for small enough values of x. This guarantees the existence of a limit as f is both monotone and bounded on near 0.

Observe that

$$\frac{d}{dx} f(x) = p x^{p-1} \ln(x) + x^p (1/x) = x^{p-1}(p \ln(x) + 1) < 0$$

if $0 < x < \mathrm{e}^{-p}$. Therefore $y = f(x)$ is decreasing on $(0, \mathrm{e}^{-p})$, so $f(x)$ increases as x decreases to 0. Therefore there is some $L \leq 0$ so that $f(x)$ approaches L as x decreases to 0. Now, apply the Shift and Scale principle:

$$\begin{aligned}
L &= \lim_{x\to 0^+} x^p \ln(x) \\
&= \lim_{x\to 0^+} (\mathrm{e}x)^p \ln(\mathrm{e}x) \\
&= \mathrm{e}^p \lim_{x\to 0^+} x^p (\ln(\mathrm{e}) + \ln(x)) \\
&= \mathrm{e}^p \lim_{x\to 0^+} (x^p + x^p \ln(x)) \\
&= \mathrm{e}^p (0 + L) \\
&= e^p L
\end{aligned}$$

Hence $L = \mathrm{e}^p L$. Since $\mathrm{e}^p \neq 1$ we have $L = 0$.

For example, if we consider the entropy function $h(x) = -(x \ln(x) + (1-x)\ln(1-x))$ we see that the entropy function has removable discontinuities at both 0 and 1, where function can be defined to be 0 and made continuous.

1.3 $\ln(x)/x^p$

If we recall that

$$\lim_{y\to 0^+} f(y) = \lim_{x\to\infty} f\left(\frac{1}{x}\right)$$

and $\ln(1/x) = -\ln(x)$, then we have

$$\lim_{x\to\infty} \frac{\ln(x)}{x^p} = \lim_{x\to\infty} -\left(\frac{1}{x}\right)^p \ln\left(\frac{1}{x}\right) = \lim_{y\to 0^+} -(y^p \ln(y)) = 0.$$

1.4 $x^p \exp(x^q)$ for any $p > 0$ and $q > 0$

We need not restrict ourselves to p and q positive rational numbers. Indeed, we only need to require $p > 0$ and $q > 0$. (The case $p \leq 0$ is easily seen to yield a limit of 0.)

Observe that for $x > 0$:

$$\frac{x^p}{\exp(x^q)} = \exp(p \ln(x)) \exp(-(x^q)) = \exp(p \ln(x) - (x^q)) = \exp\left(-x^q \left(1 - p\frac{\ln(x)}{x^q}\right)\right).$$

Now we see that

$$\begin{aligned}
\lim_{x\to\infty} \left(1 - p\frac{\ln(x)}{x^q}\right) &= 1 \\
\lim_{x\to\infty} -x^q \left(1 - p\frac{\ln(x)}{x^q}\right) &= -\infty \\
\lim_{x\to\infty} \exp\left(-x^q \left(1 - p\frac{\ln(x)}{x^q}\right)\right) &= 0
\end{aligned}$$