Calculus and Graphing

The goal is to use the first and second derivative of a function to provide information about the graph of $y = f(x)$.

1 Summary of Information from the Mean Value Theorems

We collect here what we have gleaned from the Mean Value Theorems. We will assume that the domain of f is an open interval I and that f' and f'' are defined everywhere on this interval.

1. If $[a, b] \subset I$ and $f'(x) < 0$ for each $x \in (a, b)$ then f is strictly decreasing on $[a, b]$.
2. If $[a, b] \subset I$ and $f'(x) \leq 0$ for each $x \in (a, b)$ then f is non-increasing on $[a, b]$.
3. If $[a, b] \subset I$ and $f'(x) > 0$ for each $x \in (a, b)$ then f is strictly increasing on $[a, b]$.
4. If $[a, b] \subset I$ and $f'(x) \geq 0$ for each $x \in (a, b)$ then f is non-decreasing on $[a, b]$.
5. If $a \in I$ and $x \in I$ then then there is some c lying between a and x so that
$$f(x) - (f(a) + f'(a)(x - a)) = \frac{1}{2} f''(c)(x - a)^2.$$
Hence if $f''(x) \geq 0$ on I then
$$f(x) \geq f(a) + f'(a)(x - a)).$$
In other words, the graph of $y = f(x)$ lies above the tangent line to its graph at each point $(a, f(a))$ and for all $x \in I$. Similarly, if $f''(x) \leq 0$ for each $x \in I$, then
$$f(x) \leq f(a) + f'(a)(x - a)),$$
that is, the graph lies under the tangent lines.
6. If $[a, b] \subset I$ and $x \in [a, b]$ then there is some $c \in (a, b)$ so that
$$f(x) - \left(f(a)\frac{x - b}{a - b} + f(b)\frac{x - a}{b - a} \right) = -\frac{1}{2} f''(c)(b - x)(x - a).$$
Recall that
$$y = f(a)\frac{x - b}{a - b} + f(b)\frac{x - a}{b - a}$$
is an equation of the line joining $(a, f(a))$ to $(b, f(b))$, so for $x \in [a, b]$ it is an equation of the chord joining $(a, f(a))$ to $(b, f(b))$. So we can conclude that if $f''(x) \geq 0$ on I, then every chord lies above the graph of $y = f(x)$ and if $f''(x) \leq 0$ then every chord lies below the graph. In the former case we say that the graph is **convex** or **concave up** and in the latter case we say that the graph is **concave** or **concave down**. It can be shown that if a graph is convex then at each point there is at least one tangent line, and that each of these tangent lines lies below the graph, and that if the graph is concave then at each point there is at least one tangent line, and that each of these tangent lines lies above the graph. It is important to note that the definitions of convex and concave do not require the function to be differentiable. For example, $f(x) = |x|$ is convex.

2 What calculus tells us about graphs of functions

Calculus allows us to decompose the graph of a function into four different types of arcs:

1. Non-decreasing, convex arcs. This is the case where $f'(x) \geq 0$ and $f''(x) > 0$ on the interval (a, b).
2. Non-decreasing, concave arcs. This is the case where $f'(x) \geq 0$ and $f''(x) < 0$ on the interval (a, b).

3. Non-increasing, convex arcs. This is the case where $f'(x) \leq 0$ and $f''(x) > 0$ on the interval (a, b).

4. Non-increasing, concave arcs. This is the case where $f'(x) \leq 0$ and $f''(x) < 0$ on the interval (a, b).

Consider the following examples.

2.1 A cubic polynomial example

Suppose that $f(x) = 2x^3 - 3x^2 + 1$, with domain all the real numbers. We have $f'(x) = 6x^2 - 6x = 6x(x-1)$ and $f''(x) = 12x - 6$.

It should be clear that

- $f'(x) > 0$ on $(-\infty, 0) \cup (1, \infty)$;
- $f'(x) < 0$ on $(0, 1)$;
- $f''(x) > 0$ on $(1/2, \infty)$;
- $f''(x) < 0$ on $(-\infty, \infty)$;

Since f is continuous its graph is made up of the following 4 arcs:

- An increasing, concave arc terminating at $(0, f(0)) = (0, 1)$.
- A decreasing, concave arc beginning at $(0, 1)$ and terminating at $(1/2, f(1/2)) = (1/2, 1/2)$;
- A decreasing, convex arc beginning at $(1/2, 1/2)$ and terminating at $(1, f(1)) = (1, 0)$.
- A increasing, convex arc beginning at $(1, 0)$.

Coupling this with what we learned in algebra about graphing we can produce a fairly accurate graph of $y = 2x^3 - 3x^2 + 1$.

2.2 $y = \exp(-x^2/2)$

Except for a multiplicative constant, this is the famous bell-shaped curve of statistics, discovered by Carl F. Gauss in his investigations of the positions of stars.

We have from the chain rule and the product rule:

$$\begin{aligned}
f(x) &= \exp\left(-\frac{x^2}{2}\right) \\
f'(x) &= \exp'\left(-\frac{x^2}{2}\right)\frac{d}{dx}\left(-\frac{x^2}{2}\right) \\
&= -x\exp\left(-\frac{x^2}{2}\right) \\
f''(x) &= -\exp\left(-\frac{x^2}{2}\right) + x^2\exp\left(-\frac{x^2}{2}\right) \\
&= (x^2-1)\exp\left(-\frac{x^2}{2}\right)
\end{aligned}$$

We see that

- $f'(x) > 0$ if $x < 0$;
- $f'(x) < 0$ if $x > 0$;
- $f''(x) > 0$ if $x \in (-\infty, -1) \cup (1, \infty)$
- $f''(x) < 0$ if $x \in (-1, 1)$.

Just as in the last example, we have four arcs:

- An increasing, convex arc terminating at $(-1, f(-1)) = (-1, \exp(-1/2)) \approx (1, 0.61)$;
- An increasing, concave arc beginning at $(-1, \exp(-1/2)$ and terminating at $(0, f(0)) = (0, 1)$.
- A decreasing, concave arc beginning at $(0, 1)$ and terminating at $(1, f(1)) = (1, \exp(1/2)) \approx (1, 0.61)$.
- A decreasing, convex arc beginning at $(1, \exp(1/2))$.

We also know that the initial and final arcs are asymptotic to the x-axis. In fact, the only thing that calculus told us there that we did not know from algebra is where the curve changed from convex to concave and back again.

2.3 The ratio of two quadratic functions

Suppose that

$$y = \frac{(x-1)(x-2)}{(x+1)(x+2)}$$

Note that since the domain of our function of x here is three separate intervals this is really three separate problems:

- The graph on $(-\infty, -2)$;
- The graph on $(-2, -1)$;
- The graph on $(-1, \infty)$.

A handy trick here is to use partial fractions to make computing the derivatives easier.

$$y = \frac{(x-1)(x-2)}{(x+1)(x+2)} = \frac{x^2-3x+2}{x^2+3x+2} = 1 - \frac{6x}{(x+1)(x+2)} = 1 + \frac{6}{x+1} - \frac{12}{x+2}$$

Therefore we have

$$\begin{aligned} y &= 1 + \frac{6}{x+1} - \frac{12}{x+2} \\ y' &= -\frac{6}{(x+1)^2} + \frac{12}{(x+2)^2} = 6\frac{x^2-2}{(x+1)^2(x+2)^2} \\ y'' &= \frac{12}{(x+1)^3} - \frac{24}{(x+2)^3} = 12\frac{(x+2)^3 - 2(x+1)^3}{(x+1)^3(x+2)^3} \end{aligned}$$

It is clear that y' changes sign only at $\pm\sqrt{2}$ and that y'' changes sign only at -1, -2, and at the unique solution to $(x+2)^3 = 2(x+1)^3$.

3 Optimization Problems

Since by drawing a graph we can determine where a function is largest or smallest, we can use calculus to solve optimization problems. In this case concavity is not an issue. Here are some examples.

3.1 Box Folding

A cardboard tray is to be formed from a sheet of cardboard measuring 30 centimeters by 20 centimeters by making a cut of length x in each corner and folding up the edges to make a box of height x. See problem 89 in Section 4.2 of

`http://www.mhhe.com/math/precalc/barnettcat7/student/olc/chap04pdf.mhtml`

for a diagram. What is the maximum possible volume of such a box?

Since the side heights are to be x, the base has dimensions $(20 - 2x)$ by $(30 - 2x)$, so we see that $0 \leq x \leq 10$, and the volume of the box, $V(x)$ is given by $V(x) = x(20 - 2x)(30 - 2x) =$

$4x(x-10)(x-15)$ for $0 \le x \le 10$. We sketch a simple graph of $y = V(x)$, taking note only of where the graph is increasing and decreasing.

$$V'(x) = 4((x-10)(x-15) + x(x-15) + x(x-10)) = 4(3x^2 - 50x + 150).$$

We see that

$$\begin{aligned} V'(x) > 0 \quad &\text{if} \quad 0 \le x < \frac{25-5\sqrt{7}}{3} \approx 3.924 \\ V'(x) < 0 \quad &\text{if} \quad \frac{25-5\sqrt{7}}{3} < x \le 10 \end{aligned}$$

Hence V is increasing, then decreasing and reaches its maximum when $x = \dfrac{25-5\sqrt{7}}{3}$, and the maximum volume is the value of V for this choice of x, which is about 1056.305896 cubic centimeters.

3.2 A constrained minimization problem

The rate at which a object cools is proportional to its surface area. Suppose we are to construct a cylinder that holds one liter. What are the dimensions of this cylinder if it is to have minimal surface area?

Let r represent the radius of the cylinder, and let h represent its height, both measured in centimeters. Then we have $1000 = \pi r^2 h$. The surface area, S, is given by

$$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \times \frac{1000}{\pi r^2} = 2\pi r^2 + 2000 r^{-1}$$

and r may be any positive real number.

We see that

$$\begin{aligned} S' &= 4\pi r - 2000 r^{-2} \\ &= 4r^{-2}\left(\pi r^3 - 500\right). \end{aligned}$$

We see that S' changes sign only at

$$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.419$$

and this change is from negative to positive, so the optimal dimensions are approximately $r = 5.419$ centimeters and $h = 10.839$ centimeters.

3.3 Snell's Law of Refraction

A basic assumption in physics is that light travels along the path from one point to another in the shortest time. Since the speed of light depends on the medium through which the light is passing, this path is will not be a straight line if the medium is changing.

Suppose then, that the light source is in a medium where the speed of light is b meters per second and is received in a medium where the speed of light is a meters per second. In addition, suppose that the interface between the media is a plane. For simplicity sake, have the source be at $Y = (0, B)$, the x axis represent the boundary between the media, and the point of reception be at $Z = (C, -A)$ where $A > 0$, $B > 0$ and $C > 0$. If the light crosses the x axis at $X = (x, 0)$ then the time T for transit is

$$T = \frac{\sqrt{B^2 + x^2}}{b} + \frac{\sqrt{A^2 + (C-x)^2}}{a}$$

and it is clear the minimum time will be achieved for some $x \in [0, C]$. Observe that

$$\begin{aligned} \frac{dT}{dx} &= \frac{1}{b}\frac{x}{\sqrt{x^2 + B^2}} - \frac{1}{a}\frac{C-x}{\sqrt{A^2 + (C-x)^2}} \\ \frac{d^2T}{dx^2} &= \frac{1}{b}\frac{B^2}{(\sqrt{x^2 + B^2})^3} + \frac{1}{a}\frac{A^2}{(\sqrt{(C-x)^2 + A^2})^3} > 0 \end{aligned}$$

Since dT/dx is negative at $x = 0$, positive at $x = C$ and has a positive derivative, it is zero for exactly one value of x, and that value of x is the unique solution of

$$\begin{aligned} \frac{1}{b}\frac{x}{\sqrt{x^2+B^2}} &= \frac{1}{a}\frac{C-x}{\sqrt{A^2+(C-x)^2}} \\ 0 &\leq x \leq C \end{aligned}$$

Now, let $U = (0,0)$, $R = (B,x)$, $V = (C,0)$ and $S = (x,-A)$. We have two right triangles: ΔYUX and ΔZVX. The angle $\angle RXY$ is called the angle of incidence and the angle $\angle ZXS$ is called the angle of refraction. Observe that

$$\begin{aligned} \sin(\angle YXR) &= \sin(\angle XYU) \\ &= \frac{x}{\sqrt{B^2+x^2}} \\ \sin(\angle ZXS) &= \sin(\angle XZV) \\ &= \frac{C-x}{\sqrt{A^2+(C-x)^2}} \end{aligned}$$

Thus the transit time is minimized when

$$\frac{\sin(\angle YXR)}{b} = \frac{\sin(\angle ZXS)}{a}.$$

This result is know as **Snell's Law of Refraction**.