Differential Equations

1 Antiderivatives

Up to this point we have primarily considered the following situation. Given an expression that defines a function, such as $f(x) = x^2$, what is the derivative, $f'(x)$? We now want to reverse the process. If you are given a function $h(x)$, can you find a function $H(x)$ so that $H'(x) = h(x)$. For example, can you find a function $H(x)$ so that $H'(x) = 2x$? Of course you can, you can find many such functions, and they all look like $H(x) = x^2 + C$, where C is any real number. In fact, on any interval on which $h(x)$ is defined, if $H'(x) = J'(x) = h(x)$ there is a unique constant C so that $J(x) = H(x) + C$. This is a consequence of the Mean Value Theorem, since $(H(x) - J(x))^{p}rime = 0$ on an interval implies that $H(x) - J(x)$ is constant on that interval.

We make some definitions. If $H'(x) = h(x)$ we say that $H(x)$ is an **antiderivative** of $h(x)$. The process of determining antiderivatives is called **antidifferentiation**. The key to finding antiderivatives is pattern matching based on recognizing the basic forms of derivatives, which we summarize below. In this table, we assume that all functions are appropriately differentiable. The symbols a and b will denote constants.

Function	An antiderivative
0	Constant
a	ax
$x^b, \quad b \neq -1$	$\frac{x^{b+1}}{b+1}$
$\frac{1}{x}$	$\ln(\lvert x\rvert)$
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$
$\sec^2 x$	$\tan(x)$
$\exp(x)$	$\exp(x)$
$\frac{1}{1+x^2}$	$\arctan(x)$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$
$\frac{1}{\sqrt{x^2+1}}$	$\operatorname{arcsinh}(x)$
$f'(g(x))g'(x)$	$f(g(x))$
$f'(x)g(x) + f(x)g'(x)$	$f(x)g(x)$
$af'(x)$	$af(x)$
$f'(x) + g'(x)$	$f(x) + g(x)$

There is no reason to assert incorrectly that you have found an antiderivative as you can and must check your proposed antiderivative by computing it derivative and confirming that you get the desired expression. This observation leads to one of the methods for antidifferentiation which is best described as guess and check. For example, suppose that we are given

$$h(x) = \frac{4}{3}x^{-11/7}.$$

Since we recognize that $h(x)$ is some sort of power function and we realize that differentiation transforms power functions into power functions, we would guess that an antiderivative of $h(x)$ should be $H(x) = ax^b$ for some choice of a and b. We then proceed to check our general form for $h(x)$:

$$\begin{aligned}
\frac{4}{3}x^{-11/7} &= h(x) \\
&= \frac{d}{dx}H(x) \\
&= \frac{d}{dx}ax^b \\
&= abx^{b-1}.
\end{aligned}$$

By comparing the first and last expression in our equations we have

$$\begin{aligned} b-1 &= -\frac{11}{7} \\ ab &= \frac{4}{3} \end{aligned}$$

The first equation quickly yields $b = -4/7$ transforming the second equation into

$$-\frac{4}{7}a = \frac{4}{3}$$

from which we conclude that $a = -7/3$. This gives

$$\begin{aligned} H(x) &= -\frac{7}{3}x^{-4/7} \\ h(x) &= -\frac{7}{3} \times \left(-\frac{4}{7}x^{-11/7}\right) = \frac{4}{3}x^{-11/7} \end{aligned}$$

With enough practice one discovers the following. If $b \neq -1$ then an antiderivative of x^b is

$$\frac{1}{b+1}x^{b+1}.$$

We already have from our table of antiderivatives that an antiderivative of x^{-1} is $\ln(|x|)$, so we can antidifferentiate any sum of multiples of power functions. For example, an antiderivative of

$$h(x) = \sqrt{x} + 3x^{-4} + 5x^{-1}$$

is

$$H(x) = \frac{1}{3/2}x^{3/2} + 3\left(\frac{1}{-3}x^{-3}\right) + 5\ln(|x|) + 12 = \frac{2}{3}x^{3/2} - x^{-3} + 5\ln(|x|) + 12.$$

1.1 Situations involving the Chain Rule

Recall that the chain rule says that if $H(x) = F(g(x))$ then

$$H'(x) = F'(g(x))g'(x)$$

for suitably differentiable functions F and g. Beginners find problems such as finding an antiderivative of

$$h(x) = \frac{2x+1}{(x^2+x+20)^3}$$

difficult. The source of the difficulty is insufficient practice in differentiation to recognize the present in the expression $h(x)$ are both the expression $g(x) = x^2 + x + 20$ and its derivative, $g^prime(x) = 2x + 1$. Once one trains oneself to look for such things, one readily sees that

$$h(x) = \frac{g'(x)}{(g(x))^3} = (g(x))^{-3}g'(x)$$

and so $h(x)$ appears to be the result of an application of the chain rule, that is $H(x) = F'(g(x))g'(x)$ where $F'(x) = x^{-3}$. Since we know how to antidifferentiate powers, we see that an antiderivative of

$$h(x) = \frac{2x+1}{(x^2+x+20)^3}$$

is

$$H(x) = \frac{1}{(x^2+x+20)^2}.$$

So an organizational tool if you suspect that the chain rule has been applied is to identify some part of the expression as $g(x)$, as we did with $x^2 + x + 20$, compute its derivative, in this case $2x + 1$, and see if this derivative, or one of its constant multiples, is a factor of the original expression. If this is the case, replace each the appropriate expressions with $g(x)$ and $g'(x)$ and see if you arrive at an expression of the form $f(g(x))g'(x)$. If you have, then you know the inner function in the composition, $g(x)$, and the **derivative** of the outer function in the composition, $f(x)$.

Here is another example:

$$h(x) = (1 + 2\exp(x))^9 \exp(x).$$

If we recognize that

$$\frac{d}{dx}(1 + 2\exp(x)) = 2\exp(x)$$

and we put $g(x) = (1 + 2\exp(x))$ then we see that

$$h(x) = \frac{1}{2}(1 + 2\exp(x))^3 \times 2\exp(x) = \frac{1}{2}(g(x))^3 g'(x)$$

so that an antiderivative of $h(x)$ has the form

$$H(x) = F((1 + 2\exp(x))$$

and

$$F'(y) = \frac{1}{2}y^3.$$

Since we know that an antiderivative of $(1/2)y^3$ is $(1/8)y^4$ we see that we can take

$$H(x) = \frac{1}{8}(1 + 2\exp(x))^4 + C$$

for any constant C.

1.2 The role of the arbitrary constant

In many instances we know more about the antiderivative. For example, suppose that the velocity of a particle at time t is $v(t) = 3t + 4$ meters per second, and that we know that at time $t = 3$ the position of the particle is 13 meters from a given reference point. Since velocity is the derivative of position, position is an antiderivative of velocity. The antiderivatives of $v(t)$ are $(3/2)t^2 + 4t + C$. We want an antiderivative $p(t)$ with the property that $p(3) = 13$, so we need

$$13 = p(3) = \frac{3}{2}(3)^2 + 4 \times 3 + C = \frac{51}{2} + C,$$

so that $C = -25/2$. Hence

$$p(t) = \frac{3}{2}t^2 + 4t - \frac{25}{2}$$

2 A geometric approach to antiderivatives

Recall that the geometric interpretation of multiplication of positive real numbers is the computation of the area of rectangles. The mean value theorem says that if $H(x)$ is a continuous antiderivative of $h(x)$ then $H(b)-H(a) = h(c)(b-a)$ for some c lying between a and b. In other words, if $h(x) \geq 0$ then its antiderivative H can be thought of an area, specifically the area between the graph of $y = h(x)$ and the horizontal axis.

Here are two really simple examples, and then we will tackle a difficult example.

Suppose $h(x) = 7$ for all x. Consider the region below the graph of $y = h(x)$, to the right of the vertical line through $(0, 0)$, above the x-axis, and to the left of the vertical line passing through $(x, 0)$ for any $x > 0$. This region is a rectangle, and it has an area of $7x$. Indeed, we can easily see that $7x$ is an antiderivative of 7.

Suppose that $h(x) = mx + b$ where $m > 0$ and $b > 0$. Consider the region bounded on the left by the vertical line through $(0, 0)$, above by $y = mx + b$, below by the x-axis and on the right by the vertical line through $(x, 0)$ for any positive number x. This region is a trapezoid whose altitude is x and whose bases have lengths b and $mx + b$, so it has an area of

$$A(x) = \frac{1}{2}(b + (mx + b))x = \frac{m}{2}x^2 + bx$$

which is indeed an antiderivative of $mx + b$.

Now a non-trivial example. Suppose that $h(x) = \sqrt{1-x^2}$. The graph of $y = \sqrt{1-x^2}$ is the upper half of the unit circle. As before, consider the region bounded above by $y = h(x)$, below by the x-axis, on the left by the vertical line through $(0,0)$ and on the right by the vertical line through $(x,0)$ for $0 \leq x \leq 1$. Let $A(x)$ denote the area of the region. It is clear that $A(0) = 0$ and $A(1) = \pi/4$, so now consider $x \in (0,1)$.

Divide the region into a sector, S, and a right triangle T by drawing a line from $(0,0)$ to $(x, \sqrt{1-x^2})$. The area of the triangle is

$$\frac{1}{2}x\sqrt{1-x^2}.$$

The area of the sector is $1/2$ of the radian measure of its central angle. This angle is the same as the angle $\angle ABC$ where $A = (0,0)$, $B = (x, \sqrt{1-x^2})$ and $C = (x,0)$, and the sine of $\angle ABC$ is x. Hence the area of the sector is

$$\frac{1}{2}\arcsin(x).$$

Hence

$$A(x) = \frac{1}{2}\left(x\sqrt{1-x^2} + \arcsin(x)\right).$$

It is clear that this formula even holds at $x = 0$ and $x = 1$. The claim is that the derivative of $A(x)$ is $\sqrt{1-x^2}$ for $x \in (0,1)$. To show that we are correct, we have at least three choices:

- First figure out the derivative of $\arcsin(x)$ and then apply our theorems about derivatives of products, sums, etc.
- Directly show that the derivative of $A(x)$ is $\sqrt{1-x^2}$, probably with a pinching argument.
- Prove a general result about area under curve functions.

We shall try the first two approaches. We will begin with the second one, because oddly enough, it is the simplest.

Suppose first that $0 \leq a < a+h \leq 1$. It is clear that since $\sqrt{1-x^2}$ is a decreasing function of x that

$$h\sqrt{1-(a+h)^2} \leq A(a+h) - A(a) \leq h\sqrt{1-a^2}$$

by considering rectangles drawn inside and outside the region bounded above by the semicircle, below by the x-axis, to the left by the line $x = a$ and to the right by the line $x = a+h$. Dividing this double inequality through by $h > 0$ and applying the pinching theorem we get

$$\lim_{h\to 0^+} \frac{A(a+h) - A(a)}{h} = \sqrt{1-a^2}.$$

If we have $0 \leq a+h < a \leq 1$ then we have $h < 0$ and

$$-h\sqrt{1-(a+h)^2} \geq A(a) - A(a+h) \geq -h\sqrt{1-a^2}.$$

Dividing by $-h > 0$ we get

$$\sqrt{1-(a+h)^2} \geq \frac{A(a+h) - A(a)}{h} \geq \sqrt{1-a^2}.$$

Now the pinching theorem yields

$$\lim_{h\to 0^-} \frac{A(a+h) - A(a)}{h} = \sqrt{1-a^2}.$$

Thus we have shown that for $a \in (0,1)$, $A'(a) = \sqrt{1-a^2}$ as we wished to do.

It should be clear that this demonstration could be applied to the area function for any non-negative monontone continuous function. What is remarkable is that there is a general construction that works for all continuous functions. This is the subject of the final considerations of this course, and is why we are skipping the third alternative listed above.

Note that the construction above shows that arcsin is differentiable on $(0, 1)$. Since arcsin is an odd function, we also know that it is differentiable on $(-1, 0)$. Our remaining alternative was to show that arcsin is differentiable on $(-1, 1)$. We shall do this now in two steps. First we will show that arcsin is differentiable at 0, and then show that it is differentiable for $a \in (0, 1)$. Finally we will show it is differentiable for $a \in (-1, a)$.

We begin by showing that

$$\lim_{u \to 0^+} \frac{\arcsin(u)}{u} = 1.$$

Observe that for each $u \in (0, \pi/2)$ there is a unique $h \in (0, 1)$ so that $\arcsin(u) = h$, and that for this h, $u = \sin(h)$. According to the mean value theorem, there is some $c \in (0, h)$ so that

$$\sin(h) = \cos(c)h.$$

In other words,

$$\begin{aligned} u &= \cos(c) \arcsin(u) \\ \frac{1}{\cos(c)} &= \frac{\arcsin(u)}{u}. \end{aligned}$$

Since $0 < c < h$ we know that

$$1 > \cos(c) > \cos(h) = \cos(\arcsin(u)) = \sqrt{1 - u^2}.$$

Therefore

$$\begin{aligned} 1 &< \frac{1}{\cos(c)} < \frac{1}{\sqrt{1 - u^2}} \\ 1 &< \frac{\arcsin(u)}{u} < \frac{1}{\sqrt{1 - u^2}} \end{aligned}$$

so it follows from the Pinching Principle that

$$\lim_{u \to 0^+} \frac{\arcsin(u)}{u} = 1.$$

Now, since arcsin is an odd function,

$$\lim_{u \to 0^-} \frac{\arcsin(u)}{u} = \lim_{u \to 0^+} \frac{\arcsin(-u)}{-u} = \lim_{u \to 0^+} \frac{\arcsin(u)}{u} = 1$$

so $\arcsin'(0) = 1$.

Suppose next that $a \in (0, 1)$. We need to show that

$$\frac{\arcsin(a + h) - \arcsin(a)}{h}$$

has a limit as h approaches 0. Well, for each $a \neq a + h$ in $(0, 1)$ there are numbers $u \neq v$ in $(0, \pi/2)$ so that

$$\begin{aligned} \arcsin(a) &= u \\ \arcsin(a + h) &= v \end{aligned}$$

For each such pair u and v there is some c lying between them so that

$$\begin{aligned} \sin(v) - \sin(u) &= \cos(c)(u - v) \\ \frac{1}{\cos(c)} &= \frac{v - u}{\sin(v) - \sin(u)} \\ &= \frac{\arcsin(a + h) - \arcsin(a)}{h}. \end{aligned}$$

So we can be specific in what follows we will consider the the cases of $h > 0$ and $h < 0$ separately:

$h > 0$: If $h > 0$ then $u < c < v$ so we have the following inequalities:

$$\begin{aligned}
\cos(v) &< \cos(c) < \cos(u) \\
\cos(\arcsin(a+h)) &< \cos(c) < \cos(\arcsin(a)) \\
\sqrt{1-(a+h)^2} &< \cos(c) < \sqrt{1-a^2} \\
\frac{1}{\sqrt{1-(a+h)^2}} &> \frac{1}{\cos(c)} > \frac{1}{\sqrt{1-a^2}} \\
\frac{1}{\sqrt{1-(a+h)^2}} &> \frac{\arcsin(a+h)-\arcsin(a)}{h} > \frac{1}{\sqrt{1-a^2}}
\end{aligned}$$

so by the Pinching Principle

$$\lim_{h\to 0^+} \frac{\arcsin(a+h)-\arcsin(a)}{h} = \frac{1}{\sqrt{1-a^2}}.$$

$h < 0$: The argument here is similar, with the only difference being that $u > v$.

$$\begin{aligned}
\cos(v) &> \cos(c) > \cos(u) \\
\cos(\arcsin(a+h)) &> \cos(c) > \cos(\arcsin(a)) \\
\sqrt{1-(a+h)^2} &> \cos(c) > \sqrt{1-a^2} \\
\frac{1}{\sqrt{1-(a+h)^2}} &< \frac{1}{\cos(c)} < \frac{1}{\sqrt{1-a^2}} \\
\frac{1}{\sqrt{1-(a+h)^2}} &< \frac{\arcsin(a+h)-\arcsin(a)}{h} < \frac{1}{\sqrt{1-a^2}}
\end{aligned}$$

so by the Pinching Principle

$$\lim_{h\to 0^-} \frac{\arcsin(a+h)-\arcsin(a)}{h} = \frac{1}{\sqrt{1-a^2}}.$$

So we have shown that for $a > 0$

$$\arcsin'(a) = \frac{1}{\sqrt{1-a^2}}.$$

Now we appeal to the following elementary but useful proposition.

Proposition 1 *Suppose that f is defined on $(-b, b)$ and differentiable on $(0, b)$. If f is even or f is odd, then f is also differentiable on $(-b, 0)$.*

Reason: Suppose that $a \in (0, b)$ and $k = -h$. First consider f an even function.

$$\frac{f(-a+h)-f(-a)}{h} = \frac{f(a-h)-f(a)}{h} = -\frac{f(a+k)-f(a)}{k}$$

so

$$\lim_{h\to 0} \frac{f(-a+h)-f(-a)}{h} = -\lim_{k\to 0} \frac{f(a+k)-f(a)}{k} = -f'(a)$$

so $f'(-a) = -f'(a)$, that is, f' is odd. Now suppose that f is odd. Then

$$\frac{f(-a+h)-f(-a)}{h} = \frac{-f(a-h)+f(a)}{h} = \frac{f(a-h)-f(a)}{-h} = \frac{f(a+k)-f(a)}{k}$$

$$\lim_{h\to 0} \frac{f(-a+h)-f(-a)}{h} = \lim_{k\to 0} \frac{f(a+k)-f(a)}{k} = f'(a),$$

so $f'(-a) = f'(a)$. In other words, f' is even. **QED**

From this proposition we see that

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}.$$

Since $\arccos(x) + \arcsin(x) = \pi/2$ we see that

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}.$$

Finally, since

$$\arctan(x) = \arcsin\left(\frac{x}{\sqrt{1+x^2}}\right)$$

we have from the chain rule that $\arctan(x)$ is differentiable. To find its derivative we use that

$$\begin{aligned} x &= \tan(\arctan(x)) \\ 1 &= \frac{d}{dx}\tan(\arctan(x)) = \sec^2(\arctan(x))\frac{d}{dx}\arctan(x) = (1+x^2)\frac{d}{dx}\arctan(x) \\ \frac{1}{1+x^2} &= \arctan'(x) \end{aligned}$$

Since $\operatorname{arccsc}(x) = \arcsin(1/x)$ and $\operatorname{arcsec}(x) = \arccos(1/x) = (\pi/2) - \arcsin(1/x)$ we can derive formulae for their derivatives as well. We will leave this as a homework exercise.

3 Differential Equations

A differential equation is an equation in which the terms are known functions as well as an unknown function and/or some its derivatives. For example,

$$\begin{aligned} f'(x) &= \sin(x) \\ f''(x) + 4f(x) &= 0 \\ f''(x) + 4f(x) &= \sin(3x) \\ f'(x) &= 4(10 - f(x)) \\ f'(x) &= 5f(x) \\ f''(x) + 2f'(x) + 2f(x) &= 0 \\ f'(x) &= (4 - f(x))(3 - f(x)) \end{aligned}$$

Often $f(x)$ and its derivatives are replaced by y, y', etc. For example, each of our preceding examples might be expressed respectively as

$$\begin{aligned} y' &= \sin(x) \\ y'' + 4y &= 0 \\ y'' + 4y &= \sin(3x) \\ y' &= 4(10 - y) \\ y' &= 5y \\ y'' + 2y' + 2y &= 0 \\ y' &= (4 - y)(3 - y) \end{aligned}$$

We have already encountered and solved some examples of differential equations by guessing and checking the solutions. For example, we know that any solution to

$$y'' = -r^2 y$$

has the form $y = A\cos(rx) + B\sin(rx)$ and any solution to

$$y' = ry$$

has the form $y = A\exp(rx)$. In the following two subsections we will discuss the solution of two very common types of differential equations.

3.1 First order linear equations with constant coefficents

We will show first to solve an equation of the form

$$y' + ry = g(x)$$

for the function y in terms of r and $g(x)$ and given the value of y when $x = 0$. We will look at an example:

$$y' + 5y = 1 + \exp(3x)$$

where $y = 7$ when $x = 0$. The idea is to realize that the left hand side of the equation almost has the form of the derivative of a product, and that multiplying both sides of the equation by $\exp(5x)$ produces a left hand side that is exactly the derivative of a product, since

$$\frac{d}{dx}\left(y\exp(5x)\right) = y'\exp(5x) + y\exp(5x)5 = (y' + 5y)\exp(5x).$$

Hence our differential equation is equivalent to

$$\frac{d}{dx}\left(y\exp(5x)\right) = \exp(5x) + \exp(8x).$$

Since we know how to antidifferentiate the right hand side of this equation we have

$$y\exp(5x) = \frac{1}{5}\exp(5x) + \frac{1}{8}\exp(8x) + C.$$

Since we know $y = 7$ when $x = 0$ we have

$$7 = \frac{1}{5} + \frac{1}{8} + C$$

so $C = 267/40$. Therefore

$$y\exp(5x) = \frac{1}{5}\exp(5x) + \frac{1}{8}\exp(8x) + \frac{267}{40}.$$

Multiplying through by $\exp(-5x)$ gives

$$y = \frac{1}{5} + \frac{1}{8}\exp(3x) + \frac{267}{40}\exp(-5x).$$

To solve equations of this type: $y' + ry = g(x)$, $y = a$ when $x = 0$ we repeat these same steps:

1. Multiply the equation by $\exp(rx)$.
2. Rewrite $(y' + ry)\exp(rx)$ as $(y\exp(rx))'$ to get the equation $(y\exp(rx))' = g(x)\exp(rx)$.
3. Antidifferentiate $g(x)\exp(rx)$ to get a new function $G(x)+C$ to get a new equation $y\exp(rx) = G(x) + C$.
4. Plug in $x = 0$ and $y = a$ to get an equation for C: $a = G(0) + C$ to get the equation $y\exp(rx) = G(x) + a - G(0)$.
5. Multiply both sides of this equation by $\exp(-rx)$ to get $y = G(x)\exp(-rx) + (a - G(0))\exp(-rx)$ and simplify if possible.

Here is an example from Newton's Law of Cooling. A bottle of beer is initially at 65 degrees Celcius. It is placed in water at 5 degrees Celcius. According to Newton's Law of Cooling, the rate of change of the beer is proportional to the difference between it temperature and the temperature of its surroundings. If the water does not change temperature, find an expression for the temperature of the beer in terms of time and the proportionality constant.

Solution. Let y denote the unknown temperature. We are to assume that

$$y' = r(y - 5)$$

and $y = 65$ when $t = 0$. We can write $y' - ry = -5r$. We multiply both sides by $\exp(-rt)$ to get

$$(y \exp(-rt))' = -5r \exp(-rt)$$

We can compute the antiderivative of the right hand side, it is $5 \exp(-rt) + C$ so

$$y \exp(-rt) = 5 \exp(-rt) + C.$$

Since $y = 65$ when $t = 0$ we have $65 = 5 + C$ so $C = 60$ and

$$y \exp(-rt) = 5 \exp(-rt) + 60.$$

Solving for y we have $y = 5 + 60 \exp(rt)$. If we knew the value of y at a second time we could find r. Note that since we expect the temperature to decrease we anticipate that r will be negative.