1 Derivatives continued

Another approach to the product rule is the following. Suppose that L and W are increasing positive functions. Then

$$L(a+h)W(a+h) - L(a)W(a)$$

may be understood to be the difference between the area of a rectangle that has length $L(a)$ and width $W(a)$ and one whose length is $L(a+h)$ and whose width is $W(a+h)$. If these two rectangles have a vertex in common, then we are considering the area of an ell shaped region which may dissected into three smaller rectangles with areas

$$\begin{array}{r} L(a)(W(a+h)-W(a)) \\ W(a)(L(a+h)-L(a)) \\ (W(a+h)-W(a))(L(a+h)-L(a)) \end{array}$$

In fact, it is then easy to check that for any functions L and W, it is the case that

$$\begin{aligned} & L(a+h)W(a+h) - L(a)W(a) \\ = \;& L(a)(W(a+h)-W(a)) + W(a)(L(a+h)-L(a)) + (W(a+h)-W(a))(L(a+h)-L(a)) \end{aligned}$$

Therefore, if $h \neq 0$:

$$\begin{aligned} & \frac{L(a+h)W(a+h) - L(a)W(a)}{h} \\ = \;& L(a)\frac{W(a+h)-W(a)}{h} + W(a)\frac{L(a+h)-L(a)}{h} + h\frac{W(a+h)-W(a)}{h} \times \frac{L(a+h)-L(a)}{h} \end{aligned}$$

Hence if L and W are differentiable at a then

$$\begin{aligned} & \lim_{h\to 0} \frac{L(a+h)W(a+h) - L(a)W(a)}{h} \\ = \;& \lim_{h\to 0} L(a)\frac{W(a+h)-W(a)}{h} + \lim_{h\to 0} W(a)\frac{L(a+h)-L(a)}{h} \\ & + \lim_{h\to 0} h\frac{W(a+h)-W(a)}{h} \times \frac{L(a+h)-L(a)}{h} \\ = \;& L(a)W'(a) + W(a)L'(a) + 0 \times W'(a)L'(a) \\ = \;& L(a)W'(a) + L'(a)W(a) \end{aligned}$$

We can also differentiate quotients easily:

Theorem 1 *If $f(a) \neq 0$ and f is differentiable at a the $1/f$ is differentiable at a and*

$$(1/f)'(a) = -\frac{f'(a)}{f(a)^2}$$

Reason: The key here is that since $f(a+h)$ converges to $f(a)$ as h approaches 0, we know that for all small values of h that $f(a+h) \neq 0$ if $f(a) \neq 0$. For such small values of h we have

$$\frac{1}{h}\left(\frac{1}{f(a+h)} - \frac{1}{f(a)}\right) = -\frac{1}{h} \times \frac{f(a+h)-f(a)}{f(a)f(a+h)} = -\frac{f(a+h)-f(a)}{h} \times \frac{1}{f(a)f(a+h)}$$

and this last expression converges to $-f'(a)/f(a)^2$ as claimed. **QED**

Corollary 1 (Quotient Rule) *If f and g are differentiable at a and $g(a) \neq 0$ then*

$$(f/g)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Reason: Since

$$\frac{f(x)}{g(x)} = f(x)\frac{1}{g(x)}$$

applying the product rule and the rule for reciprocals we have

$$(f/g)'(a) = f(a)(1/g)'(a) + \frac{f'(a)}{g(a)} = -f(a)\frac{g'(a)}{g(a)^2} + \frac{f'(a)}{g(a)} = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

2 Composite Functions

If the range of a function f is contained in the domain of a function g then we may create a new function called the composition of g with f, denoted by $g \circ f$. The domain of this function is the domain of f and the value of this function at a is $g(f(a))$, that is, $(g \circ f)(a) = g(f(a))$.

Theorem 2 (The Chain Rule for Derivatives) *Suppose that f is differentiable at a and g is differentiable at $f(a)$. Then $g \circ f$ is differentiable at a and $(g \circ f)'(a) = g'(f(a)) \times f'(a)$.*

Reason: In terms of the the definition of derivative we are saying the following. Suppose that

$$\begin{aligned} \lim_{x \to a} \frac{f(x) - f(a)}{x - a} &= L \\ \lim_{y \to f(a)} \frac{g(y) - g(f(a))}{y - f(a)} &= M \end{aligned}$$

Then

$$\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} = LM.$$

We would like to reason as follows. If we could say

$$\frac{g(f(x)) - g(f(a))}{x - a} = \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \times \frac{f(x) - f(a)}{x - a} \tag{1}$$

then by using facts about compositions of continuous functions we could get the chain rule. Unfortunately, we can only assert (1) this for those x for which $f(x) \neq f(a)$. We need to work around this problem, which we do as follows.

Define a new function G by

$$G(y) = \begin{cases} \dfrac{g(y) - g(f(a))}{y - f(a)} & \text{if} \quad y \neq f(a) \\ M & \text{if} \quad y = f(a) \end{cases}$$

Our assumption is that $G(y)$ is continuous at $y = f(a)$. Now define another function $H(x) = G(f(x))$. Since $f(x)$ is continuous at $x = a$ and $G(y)$ is continuous at $y = f(a)$ we know that $H(x)$ is continuous at $x = a$ and $H(a) = M$. We are done once we believe that

$$H(x)\frac{f(x) - f(a)}{x - a} = \frac{g(f(x)) - g(f(a))}{x - a}$$

since the limit of the left hand side of this equation as x approaches a is $M \times L$.

Neither side of this equation is defined for $x = a$, and if $f(x) = f(a)$ but $x \neq a$ then both sides equal 0. Finally, if $x \neq a$ and $f(x) \neq f(a)$ then

$$H(x)\frac{f(x) - f(a)}{x - a} = \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \frac{f(x) - f(a)}{x - a} = \frac{g(f(x)) - g(f(a))}{x - a},$$

which wraps up the argument. **QED**

As a consequence of this application, we know that the natural exponential function, the natural logarithm function, the circular trigonometric functions and the rational power functions are all continuous.

3 A New Notation

If we want to denote the action of computing the derivative, we will use the symbol

$$\frac{d}{dx}.$$

For example we will write

$$\begin{aligned}
\frac{d}{dx}x^2 &= 2x \\
\frac{d}{dt}\exp(t) &= \exp(t) \\
\frac{d}{dy}f(y) &= f'(y)
\end{aligned}$$

and so forth. We will also use

$$\frac{df}{dx}(a)$$

as a synonym for $f'(a)$ if we are thinking of f as a function of x.

4 Using the Chain Rule

First, recall that if $f(x) = x^2$ then $f'(a) = 2a$. Using this and the Chain Rule we can get a formula for the derivative of the product of two differentiable functions. Once we show that if $f(x) = 1/x$ then $f'(a) = -1/a^2$ we will be able to differentiate quotients as well.

4.1 The Product Rule

Theorem 3 (Product Rule) *Suppose that f and g are differentiable at a. Then $(f \cdot g)'(a) = f(a)g'(a) + f'(a)g(a)$.*

Reason: First we show that $(f \cdot f)'(a) = 2f(a)f'(a)$. This follows from the Chain Rule since $(f \cdot f)(x) = (f(x))^2$ is the composition of the squaring function and f. So, let $S(y) = y^2$ and $S'(y) = 2y$, so that $(f(x))^2 = S(f(x))$ and we have

$$\frac{d}{dx}(f(x))^2 = \frac{d}{dx}S(f(x)) = S'(f(x))f'(x) = 2f(x)f'(x).$$

Now we will use the **polarization identity** again:

$$f(x)g(x) = \frac{1}{4}(f(x)+g(x))^2 - \frac{1}{4}(f(x)-g(x))^2$$

We then have

$$\begin{aligned}
\frac{d}{dx}\frac{1}{4}(f(x)+g(x))^2 &= \frac{1}{2}(f(x)+g(x))(f'(x)+g'(x)) \\
&= \frac{1}{2}(f(x)f'(x)+g(x)f'(x)+f(x)g'(x)+g(x)g'(x)) \\
\frac{d}{dx}\frac{1}{4}(f(x)-g(x))^2 &= \frac{1}{2}(f(x)-g(x))(f'(x)-g'(x)) \\
&= \frac{1}{2}(f(x)f'(x)-g(x)f'(x)-f(x)g'(x)+g(x)g'(x))
\end{aligned}$$

so

$$\begin{aligned}
\frac{d}{dx}(f(x)\cdot g(x)) &= \frac{d}{dx}\left(\frac{1}{4}(f(x)+g(x))^2 - \frac{1}{4}(f(x)-g(x))^2\right) \\
&= \frac{d}{dx}\frac{1}{4}(f(x)+g(x))^2 - \frac{d}{dx}\frac{1}{4}(f(x)-g(x))^2 \\
&= \frac{1}{2}(f(x)f'(x)+g(x)f'(x)+f(x)g'(x)+g(x)g'(x)) \\
&\quad - \frac{1}{2}(f(x)f'(x)-g(x)f'(x)-f(x)g'(x)+g(x)g'(x)) \\
&= g(x)f'(x)+f(x)g'(x) = f'(x)g(x)+f(x)g'(x)
\end{aligned}$$

For example,

$$\begin{aligned}
\frac{d}{dx}(x\ln(x)) &= 1\ln(x) + x(1/x) = \ln(x)+1 \\
\frac{d}{dx}(\sin(x)\exp(x)) &= \cos(x)\exp(x)+\sin(x)\exp(x) = \exp(x)(\cos(x)+\sin(x))
\end{aligned}$$

4.2 Quotients

Theorem 4 (Quotient Rule) *If f and g are differentiable at a and $g(a) \neq 0$ then*

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$$

Reason: We proceed in two steps. First, let $R(x) = 1/x$, the reciprocal function. Then

$$R'(a) = \lim_{x\to a} R(x) - R(a)x - a \lim_{x\to a} a - x(ax)(x-a) = \frac{-1}{a^2}$$

(This calculation should be familiar by now, and shows that the power rule holds for $r = -1$.)

Now we can apply the Chain Rule and the Product Rule together to differentiate $f(x)/g(x) = f(x) \cdot R(x)$:

$$\begin{aligned}
\frac{d}{dx}\frac{f(x)}{g(x)} &= \frac{d}{dx} f(x) \cdot R(g(x)) = f'(x)R(g(x)) + f(x)\frac{d}{dx}R(x) \\
&= \frac{f'(x)}{g(x)} + f(x)R'(g(x))g'(x) = \frac{f'(x)}{g(x)} + f(x)\frac{-1}{(g(x))^2}g'(x) \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}
\end{aligned}$$

For example

$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\frac{\sin(x)}{\cos(x)} = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x).$$