

# Derivatives

We have considered many examples of limits of expressions of the form

$$\frac{f(x) - f(a)}{x - a}$$
$$\frac{f(a + h) - f(a)}{h}$$

as  $x$  approaches  $a$  in the former case and as  $h$  approaches 0 in the latter case. For a given function  $f$  and a given value  $a$  we know that the existence of one of these limits implies the existence of the other. We make the following definition:

Suppose that the limit as  $x$  approaches  $a$  of the expression  $(f(x) - f(a))/(x - a)$  exists. This limit is called the **derivative of  $f$  at  $a$**  and will be denoted by  $f'(a)$ . If  $f$  has a derivative at  $a$  we will say that  $f$  is **differentiable at  $a$** . In terms of limit notation:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$
$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

Note also that to say that  $f$  has a derivative at  $a$  is the same thing as saying that the function whose rule is  $(f(x) - f(a))/(x - a)$  has a removable discontinuity at  $x = a$ .

We may also write

$$\frac{df}{dx}(a)$$

in place of  $f'(a)$ .

We have already investigated the derivative of many functions. We will list them here for reference.

**Power functions:** If  $f(x) = x^r$ ,  $x > 0$  and  $r$  a positive integer or the reciprocal of a positive integer, then  $f'(a) = ra^{r-1}$  if  $a^{r-1}$  is defined. In either case this follows from the formula

$$A^n - B^n = (A - B)(A^{n-1} + A^{n-2}B + \dots + AB^{n-2} + B^{n-1}),$$

valid for any positive integer  $n$ .

**Sine and cosine:** We have seen that

- $\sin'(a) = \cos(a)$ ;
- $\cos'(a) = -\sin(a)$ .

These follow from the identities

$$\begin{aligned}\sin(a + h) &= \sin(a)\cos(h) + \cos(a)\sin(h) \\ \cos(a + h) &= \cos(a)\cos(h) - \sin(a)\sin(h)\end{aligned}$$

and the fundamental limits

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{\sin(h)}{h} &= 0 \\ \lim_{h \rightarrow 0} \frac{1 - \cos(h)}{h} &= 0\end{aligned}$$

**Log and exponential functions:** We have also seen that

- $\exp'(a) = \exp(a)$  since

$$\exp(a + h) - \exp(a) = \exp(a)(\exp(h) - 1)$$

and

$$\lim_{h \rightarrow 0} \frac{\exp(h) - 1}{h} = 1$$

- $\ln'(a) = a^{-1}$  if  $a > 0$ . This is a bit trickier and uses the shift and scale principle:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\ln(a+h) - \ln(a)}{h} &= \lim_{h \rightarrow 0} \frac{\ln(1 + (h/a))}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{a} \frac{\ln(1 + (h/a))}{(h/a)} \\ &= \frac{1}{a} \times 1 \end{aligned}$$

It is clear from our theorems about limits of sums that

**Theorem 1 (Sum Formula)** *If  $f$  and  $g$  are differentiable at  $a$  and  $c$  is a constant then the derivative of  $f(x) + cg(x)$  at  $x = a$  is  $f'(a) + cg'(a)$ .*

For example we can differentiate any polynomial:

$$\begin{aligned} f(x) &= 2x^2 - 3x + 2 & f'(a) &= 4a - 3 \\ f(x) &= 5x^3 - 7x^2 + 2 & f'(a) &= 15a^2 - 14a \end{aligned}$$

Recall also that we have seen that if  $f$  is differentiable at  $a$  then  $f$  is continuous at  $a$  since

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} \left( f(a) + (x-a) \frac{f(x) - f(a)}{x-a} \right) = f(a) + 0 \times f'(a) = f(a).$$

Hence, if  $f$  is differentiable at  $a$  then so is  $f^2$  since

$$\lim_{x \rightarrow a} \frac{(f(x))^2 - (f(a))^2}{x-a} = \lim_{x \rightarrow a} (f(x) + f(a)) \frac{f(x) - f(a)}{x-a} = 2f(a)f'(a).$$

Since

$$f(x)g(x) = \frac{1}{2} \left( (f(x) + g(x))^2 - f(x)^2 - g(x)^2 \right)$$

we have the following theorem:

**Theorem 2 (Product Rule)** *If  $f$  and  $g$  are differentiable at  $x = a$  then so is  $fg$  and*

$$(fg)'(a) = f(a)g'(a) + f'(a)g(a).$$

To see this we apply the rule for squares and for sums:

$$\begin{aligned} (fg)'(a) &= \frac{1}{2} (2(f(a) + g(a))(f'(a) + g'(a)) - 2f(a)f'(a) - 2g(a)g'(a)) \\ &= f(a)g'(a) + f'(a)g(a) \end{aligned}$$