1 The Interpretations of the Derivative

There are three major interpretations of the derivative.

1.1 The instantaneous rate of change of a function

Since the variable t brings to mind time, suppose that $Q(t)$ represents a quantity that varies with time. This could be position, velocity, momentum, mass, volume, temperature, or a myriad of other possibilities. The expression

$$\frac{Q(t) - Q(a)}{t - a}$$

represents the average rate of change of Q between times t and a. Thus $Q'(a)$ is considered to be the **instantaneous rate of change of** Q. In physics, the following instantaneous rates of change have special significance:

Velocity: The instantaneous rate of change of position is called **velocity** (not speed). The magnitude of velocity is called **speed**.

Acceleration: The instantaneous rate of change of velocity is called **acceleration**.

Force: Momentum is the product of velocity and mass. The instantaneous rate of change of momentum is called **force**. It is possible for both mass and velocity to be changing at the same time, as when a rocket is burning off its fuel. For this reason we will want to know how to compute the instantaneous rate of change of the product of two functions.

1.1.1 Related Rates:

Suppose that a car is approaching a right angle intersection at 40 miles per hour. At the instant that the car is 1/10 of a mile from the interesection, a police officer stationed 1/20 of a mile from the intersection, but on the cross street takes a radar reading on this car. What does the radar reading show the car's speed to be? The solution is the following.

Let $h(t)$ be the distance from the officer to the car at time t, measured in miles, and let $c(t)$ be the distance from the car to the intersection, also at time t and also measured in miles. Let T be the time that the car is 1/10 of a mile from the intersection, that is $C(T) = 1/10$. We want to know $h'(T)$. From the Pythagorean Theorem we know that at all times near to T,

$$(h(t))^2 = \left(\frac{1}{20}\right)^2 + (c(t))^2 \tag{1}$$

so in particular

$$(h(T))^2 = \left(\frac{1}{20}\right)^2 + (c(T))^2 = \left(\frac{1}{20}\right)^2 + \left(\frac{1}{10}\right)^2 = \frac{5}{400}$$

so $h(T) = \sqrt{5}/20$. On the other hand, from the chain rule, differentiating both sides of (1) give us $2h(t)h'(t) = 0 + 2c(t)c'(t)$. Substituting $t = T$ we get

$$2\frac{\sqrt{5}}{20}h'(T) = 2\frac{1}{10} \times 40 = \frac{1}{2}$$

which gives us

$$h'(T) = 40 \times \frac{2}{\sqrt{5}} \approx 35.78.$$

So, his radar says we are going about 4.32 miles per hour slower than we really are. So if the speed limit is 30 miles per hour, we are speeding according to his radar, but the radar does not indicate that we are 10 miles per hour over the limit.

So, can the radar be calibrated to account for this? Let us look at the problem in general. Assume that the roads intersect at a right angle, and that our speed is A miles per hour when we are a miles from the intersection, and that the officer is b miles from the intersection on the cross street. Let h denote the distance from the officer to us, and let H be the speed shown on the radar. We assume that the officer is not moving, so b is a constant.

We have

$$\begin{aligned} h^2 &= a^2 + b^2 \\ h &= \sqrt{a^2+b^2} \\ 2hH &= 2aA \\ H &= A\frac{a}{\sqrt{a^2+b^2}} \\ A &= H\frac{\sqrt{a^2+b^2}}{a}. \end{aligned}$$

In other words, if the officer knows his distance from the intersection and your distance from the intersection at the time of the radar reading, he has a known constant of proportionality, $\sqrt{a^2+b^2}/a$ to use to convert the radar reading to the our speed, no matter how fast we are going. In particular, if $a = b$, the constant is $1/\sqrt{2} \approx 0.707$, so his radar gives about 70% of our actual speed if the two of us are equidistant from the intersection.

An example from meteorology. Assume that a pellet of ice is spherical. It accumulates more ice at a rate proportional to its surface area. If the radius of the ice pellet is r, then the surface area is $4\pi r^2$ and the volume is $4\pi r^3/3$. Question: If the radius of the ice pellet is 2 millimeters when the surface area is increasing at at rate of 0.01 square millimeters per minute, how fast is the volume of the pellet increasing when the radius is 2 mm?

When the radius is 2 mm, the surface area is 16π square millimeters and the volume is $32\pi/3$ cubic millimeters. Let S denote the surface area at any time, measured in square millimeters, and let V denote the volume at the corresponding time, measured in cubic millimeters. Then we have $dS/dt = 0.01$, and by the Chain Rule:

$$\begin{aligned} \frac{dS}{dt} &= 8\pi r\frac{dr}{dt} \\ \frac{dV}{dt} &= 4\pi r^2\frac{dr}{dt} = S\frac{dr}{dt} \end{aligned}$$

At the time when r is 2 mm we have $dS/dt = 0.01$ square mm per minute, so

$$\begin{aligned} \frac{1}{100} &= 8\pi \times 2\frac{dr}{dt} = 16\pi\frac{dr}{dt} \\ \frac{dV}{dt} &= 16\pi\frac{dr}{dt} \end{aligned}$$

Comparing these two equations we have $dV/dt = 0.01$ cubic millimeters per minute.

1.2 Tangent lines

If we consider the graph equation $y = f(x)$ we may interpret the derivative as the slope of the line tangent to this graph at the point $(a, f(a))$. To justify this, consider two distinct points $(x, f(x))$ and $(a, f(a))$ lying on this graph. The quantity

$$\frac{f(x) - f(a)}{x - a}$$

is the slope of the line joining the points $(x, f(x))$ and $(a, f(a))$. It seems that the closer x is to a the more that this line looks tangent to $y = f(x)$.

Further evidence that we have a correct interpretation of the derivative as slope of a tangent line comes from cases where we can compute this slope directly. We have shown on the homework that

- If $y = x^2$ then the line $y - a^2 = 2a(x - a)$ is tangent to $y = x^2$ at the point (a, a^2);
- If $y = 1/x$ for $x > a$ then the line $y - a^{-1} = -(x - a)/a^2$ is tangent to $y = 1/x$ at the point (a, a^{-1});

by showing algebraically that in both cases the graph of the curve lies over the graph of the line. In addition, if we consider the upper semi-circle $y = \sqrt{1 - x^2}$ and $-1 < a < 1$ then the equation of the line tangent to the semi-circle at the point $(a, \sqrt{1 - a^2})$ is

$$y - \sqrt{1 - a^2} = -\frac{a}{\sqrt{1 - a^2}}(x - a)$$

since we know that the tangent to a circle is perpendicular to the radius at th the point of tangency.

On the other hand we can compute the derivative in each case:

$$
\begin{aligned}
\lim_{x\to a}\frac{x^2-a^2}{x-a} &= \lim_{x\to a} x+a = 2a \\
\lim_{x\to a}\frac{(1/x)-(1/a)}{x-a} &= \lim_{x\to a}\frac{a-x}{ax(x-a)} = -\frac{1}{a^2} \\
\lim_{x\to a}\frac{\sqrt{1-x^2}-\sqrt{1-a^2}}{x-a} &= \lim_{x\to a}\frac{(1-x^2)-(1-a^2)}{(\sqrt{1-x^2}+\sqrt{1-a^2})(x-a)} \\
&= \lim_{x\to a}\frac{a^2-x^2}{(\sqrt{1-x^2}+\sqrt{1-a^2})(x-a)} \\
&= \lim_{x\to a} -\frac{x+a}{\sqrt{1-x^2}+\sqrt{1-a^2}} = -\frac{a}{\sqrt{1-a^2}}
\end{aligned}
$$

and in each case we get the slope of the tangent line that we had computed earlier.

1.3 Best linear approximation

We say that a function of the form $L(x) = m(x-a) + f(a)$ is a linear approximation of $f(x)$ near the point $x = a$. We recognize that the graph of $y = L(x)$ is the graph of a line passing through the point $(a, f(a)$. If we want to measure how good this approximation is, one way to do is to measure the relative error we make by using this approximation when x is near to a:

$$\frac{f(x)-L(x)}{x-a}$$

and see how this relative error behaves as x approaches a. In the case where f is differentiable at a we have

$$\lim_{x\to a}\frac{f(x)-L(x)}{x-a} = \lim_{x\to a}\frac{f(x)-m(x-a)-f(a)}{x-a} = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}-m\right) = f'(a)-m.$$

It is clear then that if $m = f'(a)$ then the limiting relative error is 0. For this reason we say that the the linear function $L(x) = f'(a)(x-a) + f(a)$ is the best linear approximation of f near a. Geometrically this is saying that the tangent line is the line which best approximates the graph of $y = f(x)$ near to $(a, f(a))$. For example, if $f(x) = \sin(x)$ then the best linear approximation to $\sin(x)$ near to $x = \pi/6$ is

$$L(x) = \cos(\pi/6)\left(x-\frac{\pi}{6}\right) + \sin(\pi/6) = \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right) + \frac{1}{2}.$$

As another example, recall that if $f(x) = x^{1/2}$ then $f'(a) = (1/2)a^{-1/2}$. Therefore the best linear approximation to $\sqrt{x}$ near $x = 4$ is

$$L(x) = f'(4)(x-4) + \sqrt{4} = \frac{1}{4}(x-4) + 2.$$

For example, a good approximation to $\sqrt{4.01}$ would be

$$L(4.01) = \frac{1}{4}(4.01-4) + 2 = 2.0025.$$

In fact, $(2.0025)^2 = 4.01000625$ and to ten decimal places, $\sqrt{4.01} \approx 2.002498439$.

Using this idea with $f(x) = (1+x)^r$ for any rational number r we see that

$$(1+h)^r = f(h) \approx f(1) + f'(1)h = 1 + rh$$

is a good rule of thumb estimate, and that

$$\ln(1+h) \approx \ln(1) + (1/1)h = h.$$

1.3.1 The birthday problem

How many people do you need to have in a group so that the probability that at least two of them have the same birthday is about equal to 1/2? Ignore leapday.

If you have $1 < N < 366$ people, there are 365^N sequences of birthdays. The number of sequences that have no repetitions is $365(365-1)\cdots(365-(N-1))$ so the probability of no common birthdays is

$$P_N := \frac{365(365-1)\cdots(365-(N-1)}{365^N} = \left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\left(1-\frac{3}{365}\right)\cdots\left(1-\frac{N-1}{365}\right)$$

If we look at $\ln(P_N)$ we have

$$\begin{aligned}\ln(P_N) &= \ln\left(1-\frac{1}{365}\right)+\ln\left(1-\frac{2}{365}\right)+\ln\left(1-\frac{3}{365}\right)+\cdots+\left(1-\frac{N-1}{365}\right)\\ &\approx -\left(\frac{1}{365}+\frac{2}{365}+\frac{3}{365}+\cdots+\frac{N-1}{365}\right)\\ &= -\frac{(N-1)N}{2\cdot 365}\end{aligned}$$

so we need to solve

$$-\ln(2) = \ln(1/2) = -\frac{(N-1)N}{2\cdot 365} = -\frac{(N-1)N}{730}$$

for $N > 0$, giving us $N \approx 23$. In fact, $P_{23} \approx 0.493$, so if there are at least 23 people present who are not born on leap day, we have a little better than a 50% chance of there being a duplicate birthday.

2 The derivative as a new function, and the second derivative

It has probably occured to you, or at least to your subconscious, that given a function f we have created a new function f' by the rule

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

For example, if $f(x) = 4x^3 + 2x^2 - 9x + 11$ then $f'(x) = 12x^2 + 4x - 9$. It may be the case, as it is in this example, that f' itself has a derivative. We call the derivative of the derivative the **second derivative**, and denote it either with two primes, f'', or by

$$\frac{d^2 f}{dx^2}(x).$$

2.1 Application: Hooke's law and Acceleration

The natural application of the second derivative is in physics. If $p(t)$ is the position of an object at time t, $v(t)$ is its velocity and $a(t)$ is its acceleration then we have

$$\begin{aligned}v(t) &= p'(t)\\ a(t) &= v'(t)\end{aligned}$$

from which it is clear that $a(t) = p''(t)$, that is, the acceleration is the second derivative of the position. From this we see, for example, that if $p(t) = -4.9t^2 + 30t + 12$ meters at time t seconds, then

$$\begin{aligned}v(t) &= p'(t) = -9.8t + 30 \text{ meters per second}\\ a(t) &= v'(t) = p''(t) = -9.8 \text{ meters per second per second}\end{aligned}$$

Hooke's Law says that the force exerted by a spring that has been stretched a distance $p(t)$ is proportional to $p(t)$. Since this force acts in the direction opposite to $p(t)$, we write this as

$$F_{\text{spring}}(t) = -kp(t) \tag{2}$$

where $k > 0$ is called the **spring constant**. If the units for force are newtons and the units for distance are meters, then the units for k are newtons per meter. Since we know that for a constant mass, force is mass times acceleration we may write (2) as

$$p''(t) = -\frac{k}{m}p(t). \tag{3}$$

Now, it is clear that oscillatory behavior can be described by the sine and cosine functions, and we know that

$$\begin{aligned} \sin''(t) &= \cos'(t) = -\sin(t) \\ \cos''(t) &= -\sin'(t) = -\cos(t) \end{aligned}$$

so it should come as no surprise that if we define

$$p(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin\left(\sqrt{\frac{k}{m}}t\right)$$

where A and B are constants, then

$$\begin{aligned} p'(t) &= \frac{d}{dt}p(t) \\ &= \frac{d}{dt}\left(A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin\left(\sqrt{\frac{k}{m}}t\right)\right) \\ &= \frac{d}{dt}\left(A\cos\left(\sqrt{\frac{k}{m}}t\right)\right) + \frac{d}{dt}\left(B\sin\left(\sqrt{\frac{k}{m}}t\right)\right) \\ &= A\frac{d}{dt}\left(\cos\left(\sqrt{\frac{k}{m}}t\right)\right) + \frac{d}{dt}\left(B\sin\left(\sqrt{\frac{k}{m}}t\right)\right) \\ &= -A\sin\left(\sqrt{\frac{k}{m}}t\right)\sqrt{\frac{k}{m}} + B\cos\left(\sqrt{\frac{k}{m}}t\right)\sqrt{\frac{k}{m}} \\ &= -A\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}t\right) + B\sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}t\right) \end{aligned}$$

and

$$\begin{aligned} p''(t) &= \frac{d}{dt}p'(t) \\ &= \frac{d}{dt}\left(-A\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}t\right) + B\sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}t\right)\right) \\ &= -A\sqrt{\frac{k}{m}}\frac{d}{dt}\sin\left(\sqrt{\frac{k}{m}}t\right) + B\sqrt{\frac{k}{m}}\frac{d}{dt}\cos\left(\sqrt{\frac{k}{m}}t\right) \\ &= -A\sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}t\right)\sqrt{\frac{k}{m}} + -B\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}t\right)\sqrt{\frac{k}{m}} \\ &= -A\frac{k}{m}\cos\left(\sqrt{\frac{k}{m}}t\right) + -B\frac{k}{m}\sin\left(\sqrt{\frac{k}{m}}t\right) \\ &= -\frac{k}{m}p(t). \end{aligned}$$

For example, suppose we know that $k = 8$ Newtons per meter, and $m = 2$ kilograms. In addition, we have $p(0) = 3$ meters and $p'(0) = 7$ meters per second. Then

$$\begin{aligned} p''(t) &= -4p(t) \\ p(0) &= 3 \\ p'(0) &= 7 \end{aligned}$$

If we have any function of t of the form $f(at)$ where a is a constant, it follows from the chain rule that each application of differentiation produces a factor of a. Since $4 = 2^2$, it would appear that we want to have

$$p(t) = A\cos(2t) + B\sin(2t)$$

for some choice of A and B. To see if we are right, note that

$$\begin{aligned} p'(t) &= -A\sin(2t)\cdot 2 + B\cos(2t)\cdot 2 = -2A\sin(2t) + 2B\cos(2t) \\ p''(t) &= -2A\cos(2t)\cdot 2 - 2B\sin(2t)\cdot 2 = -4A\cos(2t) - 4B\sin(2t) = -4p(t) \end{aligned}$$

as desired. Now, to determine A and B notice that

$$\begin{aligned} 3 &= p(0) = A\cos(0) + B\sin(0) = A \\ 7 &= p'(0) = -2A\sin(0) + 2B\cos(0) = 2B \end{aligned}$$

so $A = 3$, $B = 7/2$ and

$$p(t) = 3\cos(2t) + \frac{7}{2}\sin(2t).$$

3 Implicit Differentiation

Consider the equation

$$(x^2 - y^2 - 1)(x - y)(x + y) = 0.$$

This is satisfied by any point (a, b) lying on one of the following graphs:

- The circle with radius 1 centered at the origin: $x^2 + y^2 = 1$;
- The line with slope one passing through the origin: $y = x$;
- The line with slope negative one passing through the origin: $y = -x$.

The graph of this equation is the union of this circle and the two lines, and is, therefore, not the graph of a function (check this via the vertical line test). It is, however, made of graphs of the following four functions:

- $a(x) = x$, x any real number;
- $b(x) = -x$, x any real number;
- $c(x) = \sqrt{1 - x^2}$, $-1 \leq x \leq 1$;
- $d(x) = -\sqrt{1 - x^2}$, $-1 \leq x \leq 1$.

and it seems reasonable to ask that any any point (a, b) that belongs to exactly one of these functions that there be a derivative. For example, $(1, 1)$ lies only on the graph of $y = a(x) = x$ and the derivative at $x = 1$ is 1.

Now, all of this seems quite clear because we have nicely factored the expression giving our equation. However, if the expression were multiplied out to give

$$x^4 - y^4 - x^2 + y^2 = 0$$

it would be hard to see what is happening graphically.

The method of **implicit differentiation** tells us how to find the derivative in such situations (if there is a derivative to be found). The idea is to assume that y is a differentiable function of x and apply the chain rule, quotient rule, product rule, and so forth, to compute the derivative of both the left and right sides of the equation in question as if x were the independent variable. In the case at hand we get

$$4x^3 - 4y^3y' - 2x + 2yy' = 0 \tag{4}$$

Now, if we have a point in mind, for example $(x, y) = (1, 1)$, we substitute these values into (4) and get an equation for y':

$$4 - 4y' - 2 + 2y' = 0$$

so $y' = 1$. Note that if we pick a point that is on the graphs of two or more functions at the same time, such as $(1/\sqrt{2}, 1/\sqrt{2})$ then we get no information about y':

$$\frac{4}{2\sqrt{2}} - \frac{4}{2\sqrt{2}}y' - \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}y' = 0$$

which reduces to $0 = 0$, or an nonsensical equation, such as when we pick the point $(1, 0)$:

$$4 - 0y' - 2 + 0y' = 0$$

which says 2 and 0 are equal.

Here is another example where it is not too hard to see the graph: $y^2 = (x+1)(x-1)^2$, or, $y^2 - (x+1)(x-1)^2 = 0$. It is not hard to see that $x \geq -1$ is required here. At the point $(3, -4)$ we can find y':

$$\begin{aligned} 2yy' - (x-1)^2 - 2(x+1)(x-1) &= 0 \\ -8y' - 4 - 16 &= 0 \end{aligned}$$

so $y' = -2/5$.

An example from College Algebra. You learned that any equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ had a graph that was either empty, a point, one or two lines or a conic section. We can now find the equations of tangent lines to such graphs using implicit differentiation. Consider, for example, $4x^2 + 3xy + 8y^2 - 2x - 5y - 8 = 0$. Let us find the equation of the tangent line at $(1, 1)$. We have

$$\begin{aligned} 8x + 3y + 3xy' + 16yy' - 2 - 5y' &= 0 \\ 8 + 3 + 3y' + 16y' - 2 - 5y' &= 0 \end{aligned}$$

so $9 + 14y' = 0$, giving us $y' = -9/14$. This means an equation for the tangent line at $(1, 1)$ is $y - 1 = -(9/14)(x - 1)$.

An example from Trigonometry. If $y = \arctan(x)$ then $x = \tan(y)$. Differentiating both sides of the latter relation with respect to x gives

$$1 = \sec^2(y)y' = (1 + \tan^2(y))y' = (1 + x^2)y^prime.$$

Since $y = \arctan(x)$ then y' is the derivative of $\arctan(x)$, so we have

$$\frac{d}{dx}\arctan(x) = y' = \frac{1}{1 + x^2}$$

assuming that arctan has a derivative.