Properties of Continuous Functions

We now will establish some important properties of continuous functions. What you should keep in mind is that in many regards continuous functions behave like polynomials.

1 Limits of compositions of functions

We begin by generalizing the Shift and Scale theorem. In this theorem we may have a a number or $a = \infty$ or $a = -\infty$. The same is true about L.

Theorem 1 (Composition) *Suppose that*

$$\begin{aligned} \lim_{x \to a} f(x) &= L \\ \lim_{x \to L} g(x) &= M \end{aligned}$$

Then

$$\lim_{x \to a} g(f(x)) = M.$$

Reason: We will address the case where a and L are numbers. The other cases are similar. We have to show that for each $t > 0$ there is some $d(t) > 0$ so that if $0 < |x-a| < d(t)$ then $|g(f(x)) - M| < t$. We know that there is some $b(t) > 0$ so that if $0 < |y - L| < b(t)$ then $|g(y) - M| < t$. In turn we know that there is some $c(b(t)) > 0$ so that if $0 < |x-a| < c(b(t))$ then $|f(x) - L| < b(t)$. Therefore, if $0 < |x-a| < c(b(t))$ then $|g(f(x)) - M| < t$ as required.

For example, since $|x| = \sqrt{x^2}$ and both the squareroot and squaring functions are continuous, so is the absolute value function. In turn, if f is continuous at $x = a$, so is $|f|$.

A special kind of composition is when we compose an infinite sequence with a strictly increasing infinite sequence whose range is contained in the non-negative integers. Specifically, if $k(n)$ is a strictly increasing infinite sequence whose range is contained in $\{0, 1, \ldots\}$ and $f(n)$ is another infinite sequence, we say that the infinite sequence $f(k(n))$ is a **subsequence of** $f(n)$. For example, we may take $k(n) = 2n$ or n^2. The important property of infinite sequences is this companion to the Monontone function theorem:

Theorem 2 (Convergent Subsequences) *If f is a bounded infinite sequence then it has a convergent subsequence. That is, we can find a strictly increasing infinite sequence $k(n)$ taking values in $\{0, 1, \ldots\}$ so that $f(k(n))$ is convergent. In particular, if the range of f is contained in $[a, b]$ then so is the limit of $f(k(n))$.*

Reason: We can base an argument on the Pinching Principle and the bisection idea used in College Algebra to approximate roots of polynomials.

Suppose that the range of the sequence f is contained in $[a, b]$ where $a < b$. We define several new sequences recursively. Let

- $k(0) = 0$;
- $a(0) = a$;
- $b(0) = b$ so $b(0) - a(0) = (b - a)/2^0$.
- $c(0) = (a(0) + b(0))/2$, so $c(0)$ is the midpoint of $[a(0), b(0)]$;

- $L(0) = \{n : f(n) \in [a(0), c(0)]\}$;
- $R(0) = \{n : f(n) \in [c(0), b(0)]\}$;
- $g(0) = f(k(0))$, so $a(0) \leq g(0) \leq b(0)$.

It is not possible that both $L(0)$ and $R(0)$ are bounded (and hence finite) sets, since the domain of f is all non-negative integers. If $L(0)$ is an unbounded set we set

- $k(1) = \min\{n \in L(0) : n > k(0)\}$, that is, the smallest element of $L(0)$ that is larger than $k(0)$;
- $a(1) = a(0)$
- $b(1) = c(0)$ so $b(1) < b(0)$ and $b(1) - a(1) = (b - a)/2^1$.
- $c(1) = (a(1) + b(1))/2$, so $c(1)$ is the midpoint of $[a(1), b(1)]$;
- $L(1) = \{n : f(n) \in [a(1), c(1)]\}$;
- $R(1) = \{n : f(n) \in [c(1), b(1)]\}$;
- $g(1) = f(k(1))$ so $a(1) \leq g(1) \leq b(1)$.

If $L(0)$ is a bounded set then $R(0)$ is an unbounded set and we set

- $k(1) = \min\{n \in R(0) : n > k(0)\}$, that is, the smallest element of $R(0)$ that is larger than $k(0)$;
- $a(1) = c(0)$, so $a(0) < a(1)$;
- $b(1) = b(0)$ so $b(1) - a(1) = (b - a)/2^1$.
- $c(1) = (a(1) + b(1))/2$, so $c(1)$ is the midpoint of $[a(1), b(1)]$;
- $L(1) = \{n : f(n) \in [a(1), c(1)]\}$;
- $R(1) = \{n : f(n) \in [c(1), b(1)]\}$;
- $g(1) = f(k(1))$ so $a(1) \leq g(1) \leq b(1)$.

We see that either way we have now replaced $[a, b]$ with an interval that is half as wide and that contains an unbounded part of the domain of f. By repeating this procedure over and over we produce sequences $a(n)$, $b(n)$, $k(n)$ and $g(n)$ with the following properties:

- $a(n)$ is increasing and the range of $a(n)$ is contained in $[a, b]$;
- $b(n)$ is decreasing and the range of $b(n)$ is contained in $[a, b]$;
- $b(n) - a(n) = (b - a)/2^n$;
- $k(n)$ is strictly increasing and takes values in $\{0, 1, \ldots\}$;
- $g(n) = f(k(n))$ and $a(n) \leq g(n) \leq b(n)$

If we let A and B denote the limits of $a(n)$ and $b(n)$ respectively as n tends to infinity we have that $A = B$ so $g(n)$ is convergent by the Pinching Principle. **QED**

In the next sections we will put this theorem to work to examine properties of continuous functions.

2 Continuous functions defined on closed intervals

In this section we will presume that the function f has the closed interval $[a, b]$ for its domain.

Theorem 3 (Maximum/Minimum) *If the domain of a continuous function is a closed interval* $[a, b]$ *then the range of* f *is bounded set* R. *Furthermore, there is a number* $m \in [a, b]$ *so that* $f(m)$ *is the greatest lower bound of* f, *and there is a number* $M \in [a, b]$ *so that* $f(M)$ *is the least upper bound of* R.

Reason: Suppose R is not bounded. Then for every non-negative integer n there is some $x(n) \in [a, b]$ so that $|f(x(n)| > n$. Let $y(n)$ be a convergent subsequence of $x(n)$ with limit y as n tends to infinity. On the one hand, since $|f|$ is continuous,

$$\lim_{n\to\infty} |f(y(n))| = |f(y)|.$$

On the other hand, $|f(y(n))| > n$,

$$\lim_{n\to\infty} |f(y(n))| = \infty.$$

This is a contradiction, so the R must be a bounded set.

Now, suppose that L is the least upper bound of R. For each non-negative integer n there is some $x(n) \in [a, b]$ so that $0 \le L - f(x(n)) < 1/n$. $x(n)$ is a bounded sequence. Let $y(n)$ be a convergent subsequence of $x(n)$ with limit equal to $y \in [a, b]$. We have

$$0 < L - f(y(n)) < 1/n$$

so by the Pinching Principle we have

$$0 = \lim_{n\to\infty} L - f(y_n) = L - f(y).$$

In other words, $f(y) = L$ and $y \in [a, b]$, that is, L is in the range of f.

A similar argument shows that the greatest lower bound of the range of f is also in the range of f. **QED**

Theorem 4 (Uniform Continuity) *If the domain of a continuous function* f *is a closed interval* $[a, b]$ *then for every* $t > 0$ *there is some* $d(t)$ *such that if* $a \le x \le y \le b$ *and* $0 \le y - x \le d(t)$ *then* $|f(y) - f(x)| < t$.

Reason: Suppose the theorem is false. This means that there is some $t > 0$ so that for each choice of $d(t) > 0$ there is some pair $x < y$ for which $|f(x) - f(y)| \ge t > 0$.

We choose a sequence of pairs $x(n) < y(n)$ so that

$$\begin{aligned} 0 \le y(n) - x(n) &\le 1/n \\ 0 < t &\le |f(y(n)) - f(x(n))| \end{aligned}$$

Since $a \le x(n) < y(n) \le b$ there is a strictly increasing positive integer valued infinite sequence $k(n)$ so that

$$\lim_{n\to\infty} y(k(n)) = L.$$

Since $k(n) \geq n$ we have

$$0 \leq y(k(n)) - x(k(n)) \leq 1/k(n) \leq 1/n$$

so by the Pinching Principle,

$$\lim_{n\to\infty} x(k(n)) = L$$

as well. Now we have a contradiction, as

$$0 < t \leq \lim_{n\to\infty} |f(y(k(n))) - f(x(k(n)))| = |L - L| = 0.$$

Hence the Theorem must be true! **QED**

3 Solutions of equations

In College Algebra you learned that you could solve polynomial equations approximately by using bisection. We will use that idea to establish

Theorem 5 (Interval Range) *Suppose that the domain of a continuous function f is an interval. Then the range of f is also an interval.*

To do this we consider a simpler Lemma:

Lemma 1 (Sign Changes) *Suppose that the function f is continuous on an interval I and changes sign on I. Then the is some $c \in I$ with $f(c) = 0$.*

Reason: We shall make precise the bisection method you studied in College Algebra. We first consider a simple case. Suppose that f is continuous on $[0, 1]$, that $f(0) < 0$ and $f(1) > 0$. Put $a(0) = 0$, $b(0) = 1$ and $c(0) = 1/2$. This is the bisection idea.

- If $f(c(0)) < 0$ put $a(1) = c(0) > a(0)$ and $b(1) = b(0)$.
- If $f(c(0)) \geq 0$ put $a(1) = a(0)$ and $b(1) = c(0) < b(0)$.

Observe that

- $0 = a(0) \leq a(1) \leq b(1) \leq b(0) = 1$
- $|b(n) - a(n)| = 2^{-n}$ for $n \in \{0, 1\}$.
- $f(a(n)) \leq 0 \leq f(b(n))$ for $n \in \{0, 1\}$.

Suppose we have repeated this procedure k times to get

- $0 = a(0) \leq a(1) \leq \cdots \leq a(k) \leq b(k) \leq \cdots \leq b(1) \leq b(0) = 1$
- $|b(n) - a(n)| = 2^{-n}$ for $n \in \{0, 1, \ldots, k\}$.
- $f(a(n)) \leq 0 \leq f(b(n))$ for $n \in \{0, 1, \ldots, k\}$.

Then we get $a(k + 1)$ and $b(k + 1)$ by the following rule. Let $c(k)$ be the midpoint of $[a(k), b(k)]$. Then

- If $f(c(k)) < 0$ put $a(k + 1) = c(k) > a(k)$ and $b(k + 1) = b(k)$.

- If $f(c(k)) \geq 0$ put $a(k+1) = a(k)$ and $b(k+1) = c(k) < b(k)$.

In this way we recursively define two monotone bounded infinite sequences $a(n)$ and $b(n)$. These infinite sequences are convergent, and since $b(n) - a(n) = 2^{-n}$ they have the same limit, which we denote by c, and $0 \leq c \leq 1$.

Since f is continuous on $[0,1]$ it is continuous at c. What is more, since $f(a(n)) \leq 0$ and $f(b(n)) \geq 0$ for all n,

$$\begin{aligned} f(c) &= \lim_{n\to\infty} f(a(n)) \leq 0 \\ f(c) &= \lim_{n\to\infty} f(b(n)) \geq 0 \end{aligned}$$

so $0 \leq f(c) \leq 0$, that is, $f(c) = 0$.

For the general case, suppose that $u < v$ and g is continuous on $[u,v]$ with $g(u) < 0 < g(v)$. Put $f(x) = g((1-x)u + xv)$. Then f is continuous on $[0,1]$ and $f(0) < 0$ and $f(1) > 0$. Then there is some c in $[0,1]$ so that $f(c) = 0$, and if we put $d = (1-c)u + cv$ then d is in $[u,v]$ and $g(d) = f(c) = 0$. Finally, if $g(u) > 0$ and $g(v) < 0$ then let $f(x) = -g((1-x)u + xv)$. **QED**

To get the Theorem from the Lemma, suppose that g is continuous on an interval I containing a and b, and $g(a) \neq g(b)$. Suppose that d lies strictly between $g(a)$ and $g(b)$. Put $f(x) = g(x) - d$. Then f changes sign on I, so there is some c in I so that $0 = f(c) = g(c) - d$. In other words, there is some c in I with $g(c) = d$.