Continuity

We shall now consider functions whose limits are the corresponding functional values. This collection of functions includes rational functions, trigonometric functions, and the natural exponential and logarithmic functions.

1 Definition of Continuity

We will say that the function $f(x)$ is **right continuous at** $x = a$ if

$$\lim_{x \to a^+} f(x) = f(a)$$

left continuous at $x = a$ if

$$\lim_{x \to a^-} f(x) = f(a)$$

and **continuous at** $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

If the domain of f is $[a, b]$ we will say that f is continuous at a if it is right continuous at a and that it is continuous at b if it is left continuous at b. We will say that f is continuous if f is continuous at each point in its domain.

The following is a direct consequence of the corresponding results for limits.

Theorem 1 (Algebra of Continuous Functions) *Suppose that f, g and h have a common domain including either $[a, b)$ or $(b, a]$, that each of these functions is continuous at a and that $h(a) \neq 0$. Then $f + g$, fg and f/h are all continuous at a.*

1.1 Removable discontinuities

A function f is said to have a removable discontinuity at a if

$$\lim_{x \to a} f(x) = L$$

and either a is not in the domain of f or $f(a) \neq L$. In either case we re-define f so that $f(a) = L$. With this new definition of f we have continuous at a.

For example, if $f(x) = x^2/x$ then f is not defined at 0 but

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x = 0$$

so we would redefine f to say that $f(x) = x^2/x$ for $x \neq 0$ and $f(0) = 0$.

1.2 Limits and Composite Functions

Theorem 2 (Limits of Composites) *If*

$$\begin{aligned} \lim_{x \to a} f(x) &= M \\ \lim_{x \to M} g(x) &= L \end{aligned}$$

then

$$\lim_{x \to a} g(f(x)) = L.$$

If

$$\begin{aligned} \lim_{x \to \infty} f(x) &= M \\ \lim_{x \to M} g(x) &= L \end{aligned}$$

then

$$\lim_{x \to \infty} g(f(x)) = L.$$

In particular, if f is continuous at a and g is continuous at $f(a)$ then $g(f(x))$ is continuous at a.

Reason: We consider the finite limit case first. The other case is similar and its demonstration is left to the reader.

Suppose that $t > 0$. We proceed in two steps. First there is some d_t^g so that if $0 < |y-M| < d_t^g$ then y is in the domain of g and $|g(y)-L| < t$. Second, there is some d_t^f so that if $0 < |x-a| < d_t^f$ then x is in the domain of f and $|f(x) - M| < d_t^g$. Therefore, if $0 < |x - a| < d_t^f$ then $f(x)$ is in the domain of g and $|g(f(x)) - L| < t$ as desired. **QED**

1.3 Application

Suppose that

$$\begin{aligned} \lim_{x \to a} \frac{f(x) - f(a)}{x - a} &= L \\ \lim_{y \to f(a)} \frac{g(y) - g(f(a))}{y - f(a)} &= M \end{aligned}$$

Show that $f(x)$ is continuous at $x = a$ and

$$\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} = LM.$$

Solution: Observe that if $x \neq a$ then

$$f(x) = f(a) + f(x) - f(a) = f(a) + (x - a)\frac{f(x) - f(a)}{x - a}$$

so

$$\lim_{x \to a} f(x) = \lim_{x \to a} f(a) + (x - a)\frac{f(x) - f(a)}{x - a} = f(a) + 0 \times L = f(a),$$

so $f(x)$ is continuous at $x = a$.

Next, define a new function F by

$$G(y) = \begin{cases} \dfrac{g(y) - g(f(a))}{y - f(a)} & \text{if} \quad y \neq f(a) \\ M & \text{if} \quad y = f(a) \end{cases}$$

Our assumption is that $G(y)$ is continuous at $y = f(a)$. Now define another function $H(x) = G(f(x))$. Since $f(x)$ is continuous at $x = a$ and $G(y)$ is continuous at $y = f(a)$ we know that $H(x)$ is continuous at $x = a$ and $H(a) = M$. We are done once we believe that

$$H(x)\frac{f(x) - f(a)}{x - a} = \frac{g(f(x)) - g(f(a))}{x - a}.$$

since the limit of the left hand side of this equation as x approaches a is $M \times L$.

Neither side of this equation is defined for $x = a$, and if $f(x) = f(a)$ but $x \neq a$ then both sides equal 0. Finally, if $x \neq a$ and $f(x) \neq f(a)$ then

$$H(x)\frac{f(x) - f(a)}{x - a} = \frac{g(f(x)) - g(f(a))}{f(x) - f(a)}\frac{f(x) - f(a)}{x - a} = \frac{g(f(x)) - g(f(a))}{x - a},$$

which wraps up the argument. **QED**

As a consequence of this application, we know that the natural exponential function, the natural logarithm function, the circular trigonometric functions and the rational power functions are all continuous.

2 The Bisection Method and Properties of Continuous Functions

Suppose f is a function with domain A, and $\emptyset \neq B \subset A$. We say that the function g is the **restriction of F to B** if the domain of g is B and $g(b) = f(b)$ for every $b \in B$.

Lemma 1 (Interval Lemma) *Suppose that I_0 is a bounded interval and that for each positive integer n there is a interval I_n so that $I_n \subset I_{n-1}$. Let $a(n)$ denote the left endpoint of I_n and let $b(n)$ denote the right endpoint of $I(n)$. Then there are real numbers $A \leq B$ such that*

$$\begin{aligned} \lim_{n\to\infty} a(n) &= A \\ \lim_{n\to\infty} b(n) &= B. \end{aligned}$$

Reason: The function $a(n)$ is non-decreasing and bounded above, while the function $b(n)$ is non-increasing and bounded below. **QED**

Theorem 3 (Zero Property) *Suppose that $a < b$, f is continuous on $[a,b]$ and $f(a) \leq 0 \leq f(b)$. Then there is some $c \in (0,1)$ so that $f(c) = 0$.*

Reason: We will apply the proceeding lemma and make a recursive construction. Put $I_0 = [a,b]$. Suppose that for any non-negative integer n intervals $I_0, I_1, \ldots, I_n$ have been constructed satisfying the conditions of the lemma and the additional condition that for each integer $k \in \{0, 1, \ldots, n\}$

- $f(a(k)) \leq 0 \leq f(b(k))$;
- $b(k) - a(k) = (b-a)/2^k$.

Let $c(n)$ be the midpoint of I_n. If $f(c(n)) \geq 0$ put $I_{n+1} = [a(n), c(n)]$. Otherwise put $I_{n+1} = [c(n), b(n)]$. In either case $I_{n+1} \subset I_n$, $f(a(n+1)) \leq 0 \leq f(b(n+1))$ and $b(n+1) - a(n+1) = (b-a)/2^{n+1}$. In this way we construct a sequence of intervals satisfying the Interval Lemma and with the property that for each non-negative integer n:

- $f(a(n)) \leq 0 \leq f(b(n))$;
- $b(n) - a(n) = (b-a)/2^n$.

satisfying our lemma. In addition we have $b(n) - a(n) = (1/2)^b$, so $a(n)$ and $b(n)$ have a common limit $c \in [a,b]$. Finally, since f is continuous at c, we have

$$\begin{aligned} f(c) = \lim_{n\to\infty} f(a(n)) &\leq 0 \\ f(c) = \lim_{n\to\infty} f(b(n)) &\geq 0 \end{aligned}$$

so $f(c) = 0$. **QED**

Corollary 1 (Intermediate Value Property) *If the domain of a continuous function is an interval than so is its range.*

Proof: Suppose that a and b are in the domain of g, where $a < b$. Suppose that $g(a) < d < g(b)$. We have to show there is some $c \in [a,b]$ with $g(c) = d$. Define

$$f(x) = g(x) - d.$$

Then f is continuous on $[a,b]$, and there is some $c \in [a,b]$ so that $g(c) - d = f(c) = 0$, so $g(c) = d$. If $g(a) > g(b)$ then define $f(x) = d - g(x)$. **QED**

Theorem 4 (Maximum/Minimum Property) *If the domain of a continuous function is a closed and bounded interval then so is its range.*

Reason: Suppose that $a < b$ and f is a continuous function with domain $[a,b]$. We have already established that the range of f is an interval. We must show that this interval is a bounded set and that it contains its endpoints.

First we show that f is bounded. We do this by contradiction. Suppose the range is not bounded above. We will again make a recursive construction and apply the Interval Lemma. Put $a(0) = a$, $b(0) = b$, $I_0 = [a(0), b(0)]$ and $c(0)$ the midpoint of I_0. Note that f is unbounded on on I_0. Suppose now that intervals $I_0, I_1, \ldots, I_n$ have been constructed satisfying the Interval Lemma and with the additional properties that for each $k \in \{0, 1, \ldots, n\}$

- the range of the restriction of f to I_k is unbounded;
- $b(k) - a(k) = (b - a)/2^k$.

Let $c(n)$ be the midpoint of I_n. If the range of the restriction of f to $[a(n), c(n)]$ is unbounded, put $I_{n+1} = [a(n), c(n)]$. If not, then the range of the restriction of f to $[c(n), b(n)]$ is unbounded and we put $I_{n+1} = [c(n), b(n)]$. Therefore $I_{n+1} \subset I_n$, the range of the restriction of f to I_{n+1} is unbounded, and $b(n + 1) - a(n + 1) = (b - a)/2^{n+1}$. In this way we construct recursively a sequence of intervals satisfying the Interval Lemma and with the properties that for each non-negative integer n,

- the range of the restriction of f to I_n is unbounded;
- $b(n) - a(n) = (b - a)/2^n$.

We can in this way construct recursively a sequence of intervals satisfying the Interval Lemma, and with the additional properties that the restriction of f to any one of these intervals is unbounded and that $b(n) - a(n) = (1/2)^n$. Let c be the common limit of $a(n)$ and $b(n)$:

$$\lim_{n\to\infty} a(n) = c = \lim_{n\to\infty} b(n)$$

Since f is continuous at c there is some $d > 0$ such that if x is in the domain of f and $|x - c| < d$ then $|f(x) - f(c)| < 1$. In particular, this means that the restriction of f to $(c - d, c + d)$ is bounded. However, since

$$\lim_{n\to\infty} (1/2)^n = 0$$

there is some n so that $(1/2)^n < d/2$, and for this value of n, we have $I_n \subset (c - d, c + d)$, which says that the restriction of f to $(c - d, c + d)$ is not bounded. This is our contradiction.

Now for endpoint containment. The argument is similar to the one just given to establish that the range of f is bounded. Let R_0 denote the range of f, and let $L = \text{lub}(R_0)$ and $G = \text{glb}(R_0)$. We know that $(G, L) \subset R_0 \subset [G, L]$. We want to show that $R_0 = [G, L]$.

Let $a(0) = a$, $b(0) = b$, $I_0 = [a(0), b(0)]$ and $c(0)$ be the midpoint of I_0. Suppose that n is a non-negative integer and intervals $I_0, I_1, \ldots I_n$ have been constructed satisfying the Interval Lemma and with the properties that for each $k \in \{0, 1, \ldots, n\}$

- the least upper bound of the range of the restriction of f to I_k is L;
- $b(k) - a(k) = (b - a)/2^k$

Let $c(n)$ denote the midpoint of I_n. If the least upper bound of the range of the restriction of f to $[a(n), c(n)]$ is L, let $I_{n+1} = [a(n), c(n)]$. If not, the the least upper bound of the range of the restriction of f to $[c(n), b(n)]$ must be L, and we put $I_{n+1} = [c(n), b(n)]$. We now have $I_{n+1} \subset I_n$, the least upper bound of the range of the restriction of f to I_{n+1} is L, and $b(n + 1) - a(n + 1) = (b - a)/2^{n+1}$. In this way we recursively construct a sequence of intervals satisfying the interval lemma and with the additional property that for each non-negative integer n,

- the least upper bound of the range of the restriction of f to I_n is L;
- $b(n) - a(n) = (b - a)/2^n$

Let c be the common limit of the functions $a(n)$ and $b(n)$, that is,

$$\lim_{n\to\infty} a(n) = c = \lim_{n\to\infty} b(n).$$

We claim that $f(c) = L$. We know that $f(c) \le L$ since $f(a(n)) \le L$ for each non-negative integer n, so assume that $f(c) < L$. Put $t = (L - f(c))/2$. We know that there is some $d_t > 0$ so that if x is in the domain of f and $|x - c| < d_t$ then $|f(x) - f(c)| < t$, which implies that

$$f(x) < f(c) + \frac{L - f(c)}{2} = \frac{f(c) + L}{2} < L.$$

As in our proof that the range of f is bounded, there is some n so that $I_n \subset (c - d + t, c + d_t)$. Thus we have arrived a contradiction, since we have shown that $(L + f(c))/2 < L$ is an upper bound for the range of the restriction of f to I_n.

The argument that G is in the range of f is similar and will be omitted. **QED**

Theorem 5 (Uniform Continuity) *Suppose that $a < b$ and f is continuous on its domain $[a, b]$. Then for each $t > 0$ there is some $d_t > 0$ such that if $a \le x \le y \le b$ and $|x - y| < d_t$ then $|f(x) - f(y)| < t$.*

Reason: Throughout this discussion all variables will take values in $[a, b]$ unless we say otherwise.

Suppose that $t > 0$.

For each u there is some $v_u > 0$ so that if $|x - u| < v_u$ then $|f(x) - f(v)| < t/2$. Let

$$J_u = \left(u - \frac{v_u}{2}, u + \frac{v_u}{2}\right),$$

so if $x \in J_u$ then $|f(x) - f(u)| < t/2$.

We will now show that there is a positive integer N and a finite set $\{u_1, u_2, \ldots, u_N\} \subset [a, b]$ so that

$$[a, b] \subset J_{u_1} \cup J_{u_2} \cdots \cup J_{u_N}. \tag{1}$$

We will proceed by contradiction and use an argument similar to the one used to show that the range of f is bounded.

First observe $[a, b]$ is contained in the union of all the J_u since $u \in J_u$ for each $u \in [a, b]$. Suppose then that our claim is false. Put $a(0) = a$, $b(0) = b$, $I_0 = [a(0), b(0)]$ and $c(0)$ equal to the midpoint of I_0. Suppose we have constructed intervals $I_0, I_1, \ldots, I_n$ so that the Interval Lemma holds and (1) fails for any positive integer N. If (1) fails for $[a(n), c(n)]$ put $I_{n+1} = [a(n), c(n)]$. If not, then (1) must fail for $[c(n), b(n)]$ and we put $I_{n+1} = [c(n), b(n)]$. In this way we construct recursively a sequence of intervals I_n satisfying the Interval Lemma and having the property that for each non-negative integer n

- (1) fails for I_n;
- $b(n) - a(n) = (b - a)/2^n$.

There is then some $c \in [a, b]$ with

$$\lim_{n\to\infty} a(n) = c = \lim_{n\to\infty} b(n).$$

Since the width of the intervals I_n shrinks to 0, there is some n for which $I_n \subset J_c$, a contradiction. In other words, there is some positive integer N and some set $\{u_1, u_2, \ldots, u_N\} \subset [a, b]$ so that (1) is true.

Now, put $d_t = (1/2)\min\{v_{u_1}, v_{u_2}, \ldots v_{u_N}\} > 0$, and suppose that $x < y$ and $|y - x| < d_t$. Then $x \in J_{u_k}$ for some k, so

$$\begin{aligned} |x - u_k| &< \frac{v_k}{2}, \\ |y - u_k| &= |y - x + x - u_k| \le |y - x| + |x - u_k| \le d_t + \frac{v_k}{2} \le \frac{v_k}{2} + \frac{v_k}{2} = v_k. \end{aligned}$$

Therefore,

$$|f(x) - f(y)| = |f(x) - f(u_k) + f(u_k) - f(y)| \le |f(x) - f(u_k)| + |f(y) - f(u_k)| < t/2 + t/2 = t,$$

as desired. **QED**