

The complex exponential function

1 Comment

You will not need this material in Math 231, but you will need it in later course in mathematics, physics and electrical engineering.

2 Why a complex exponential function should exist

Recall that by definition,

$$\exp(r) = \lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n$$

where r is any real number. It will be shown in Math 232 that

$$\exp(r) = \lim_{n \rightarrow \infty} \left(1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \cdots + \frac{r^n}{n!}\right).$$

(We could do this now with the binomial theorem, but this topic is not a part of this course.)

In either case it is the case that we could consider r to be a complex number and have an infinite sequence of complex numbers. In the case where $r = it$ where t is a real number and $i^2 = -1$ we would get

$$\begin{aligned} \exp(it) &= \lim_{n \rightarrow \infty} \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots + (-1)^n \frac{t^{2n}}{(2n)!}\right) \\ &\quad + i \lim_{n \rightarrow \infty} \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots + (-1)^n \frac{t^{2n-1}}{(2n-1)!}\right) \\ &= f(t) + ig(t). \end{aligned}$$

The existence of the two limits, which I have called $f(t)$ and $g(t)$ is easily established via the theorem on limits of monotone functions, and we can show that

$$\begin{aligned} f(0) &= 1 \\ g(0) &= 0 \\ f'(t) &= -g(t) \\ g'(t) &= f(t) \end{aligned}$$

all of which allows us to conclude that $f(t) = \cos(t)$ and $g(t) = \sin(t)$, as pretty remarkable result:

Theorem 1 For each real number t ,

$$\exp(it) := \lim_{n \rightarrow \infty} \left(1 + \frac{it}{n}\right)^n = \cos(t) + i \sin(t).$$

For example,

$$\exp(2\pi i) = 1.$$

It follows from the addition formulae for sine and cosine that for any two real numbers t and s that

$$\begin{aligned} \exp(it + is) &= \cos(s + t) + i \sin(s + t) \\ &= (\cos(t) \cos(s) - \sin(t) \sin(s)) + i (\sin(t) \cos(s) + \sin(s) \cos(t)) \\ &= (\cos(t) + i \sin(t)) \times (\cos(s) + i \sin(s)) \\ &= \exp(it) \exp(is) \end{aligned}$$

so the exponential function property is still valid. In fact, for any complex number $a + bi$, we may define

$$\exp(a + bi) = \exp(a) \exp(bi)$$

and we get an exponential function defined on all complex numbers. By this we mean that $\exp(x) \exp(y) = \exp(x + y)$ for x and y complex numbers, not just real numbers.

3 The relation to the geometric properties of complex numbers

Recall that we may interpret $a + bi$ as a point in the plane corresponding to the point (a, b) . From the Pythagorean Theorem and the definition of absolute value as the distance from a number to 0 we see that

$$|a + bi|^2 = a^2 + b^2 = (a + bi)(a - bi)$$

so $|a + bi| = \sqrt{a^2 + b^2} = |a - bi|$. Recall that the number $a - bi$ is called the **complex conjugate** of $a + bi$. If $|a + bi| \neq 0$ then

$$a + bi = |a + bi| \left(\frac{a}{|a + bi|} + i \frac{b}{|a + bi|} \right) = |a + bi| (\cos(\theta) + i \sin(\theta))$$

where θ is an angle measured (in radians, please) from the ray joining 0 to 1 to the ray joining 0 and $a + bi$. We usually choose $0 \leq \theta < 2\pi$, but we don't have to. Once you choose θ you can replace it by $\theta + 2n\pi$ where n is any integer.

Now that we have the complex exponential function, we see that we can write any non-zero complex number $a + bi$ as $\exp(c + i\theta)$ where θ is as above and $c = \ln(|a + bi|)$. For example,

$$1 + i = \sqrt{2} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) = \exp \left(\ln(\sqrt{2}) + i \frac{\pi}{4} \right).$$

4 Derivatives

If c is any complex number it is easy to check that

$$\frac{d}{dx} \exp(cx) = c \exp(cx)$$

by proceeding in two steps. First, write $c = a + bi$ so $\exp(cx) = \exp(ax) \exp(ibx)$. If we can differentiate the second term then we can apply the product rule.

$$\begin{aligned}\frac{d}{dx} \exp(ibx) &= \frac{d}{dx} (\cos(bx) + i \sin(bx)) \\ &= -b \sin(bx) + ib \cos(bx) \\ &= ib(i \sin(bx) + \cos(bx)) \\ &= ib \exp(bx).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{d}{dx} \exp(cx) &= \frac{d}{dx} (\exp(ax) \exp(ibx)) \\ &= a \exp(ax) \exp(ibx) + \exp(ax)(ib \exp(ibx)) \\ &= (a + bi) \exp(ax) \exp(ibx) \\ &= c \exp(cx)\end{aligned}$$

For example, if $f(x) = \exp((2 + 3i)x)$ then $f'(x) = (2 + 3i) \exp((2 + 3i)x)$.