How Installment Loan Payments Are Calculated

Suppose I want to borrow B dollars for M months at $100r\% > 0$ interest per month and pay back P dollars per month. What must P be in order to pay back the loan? This called the level payment problem. For example, I want to borrow \$50,000 for 15 years at 12% annual interest. In this case $r = 0.01$, $M = 180$, and P is to be determined.

We will see that it will be convenient to define $s = 1 + r > 1$.

Let us walk through the calculations step by step, one month at a time until we see the general pattern. We will let B_k denote what we owe immediately after we have made the k^{th} payment.

After one month, we owe B dollars plus the interest on B dollars, rB, so we owe $B + rB = (1 + r)B = sB$ dollars. We then send the bank a check for P dollars, so we owe $sB - P$ dollars. Therefore

$$B_1 = sB - P. \tag{1}$$

After two months, we owe B_1 dollars plus the interest on B_1 dollars, rB_1 dollars, so we owe $B_1 + rB_1 = (1 + r)B_1 = sB_1$ dollars. We then send the bank a check for P dollars, so we owe $sB_1 - P$ dollars. Therefore

$$\begin{aligned} B_2 &= sB_1 - P \\ &= s(sB - P) - P \quad \text{use (1)} \\ &= s^2B - P(1 + s) \qquad (2) \end{aligned}$$

After three months, we owe B_2 dollars plus the interest on B_2 dollars, rB_2 dollars, so we owe $B_2 + rB_2 = (1 + r)B_2 = sB_2$ dollars. We then send the bank a check for P dollars, so we owe $sB_2 - P$ dollars. Therefore

$$\begin{aligned} B_3 &= sB_2 - P \\ &= s(s^2B - P(1 + s)) - P \quad \text{use (2)} \\ &= s^3B - P(1 + s + s^2) \qquad (3) \end{aligned}$$

At this point, notice that we always have the following relation:

$$B_{k+1} = B_k + rB_k - P = sB_k - P,$$

and that it appears that

$$B_k = s^kB - P(1 + s + \cdots + s^{k-1}).$$

We can prove this last assertion by induction. It is clearly true when $k = 1$, 2 and 3 by our example calculations. Suppose it is true for $k = 1, 2, 3, \ldots, m$. Then

$$\begin{aligned} B_{m+1} &= sB_m - P \\ &= s(s^mB - P(1 + s + \ldots + s^{m-1})) - P \\ &= s^{m+1}B - P(1 + s + \cdots + s^m) \end{aligned}$$

which is what we wanted to prove. Now, since $s \neq 1$, we have

$$B_m = s^m B - P\frac{1-s^m}{1-s} = s^m - P\frac{(1+r)^m - 1}{r}. \tag{4}$$

Our goal is to have the M^{th} payment bring our balance to 0, that is

$$0 = B_M = s^M B - P\frac{(1+r)^M - 1}{r},$$

an equation that can be easily solved for P:

$$P = B(1+r)^M \frac{r}{(1+r)^M - 1}. \tag{5}$$

Observe that $B(1+r)^M$ is the value of B dollars on deposit for M months at $100r\%$ per month interest.

In fact, I borrowed $50,000$ dollars for 15 years at 9.75% per year. This puts $r = 0.0975/12 = .008125$ and from (5) my monthly payments for my mortgage were about (because the bank rounds to the nearest penny and makes it up on the last payment)

$$529.69 \approx 50,000(1.008125)^{180}\frac{0.008125}{(1.008125)^{180} - 1}$$

dollars and over the life of the loan I would have paid \$45342.64 in interest.

If the annual interest rate had been 6%, which is comparable to today's rates, then $r = 0.005$ and my monthly payments would have been

$$421.93 \approx 50,000(1.005)^{180}\frac{0.005}{(1.005)^{180} - 1}$$

and a total of \$25947.12 in interest would be paid over the life of the loan.

Finally, if you wanted to stretch the payments over 30 years (360 months) then at 6% your monthly payments drop to \$299.78 but you pay \$57919.10 in interest over the life of the loan.