

SECTION 10-6 Cramer's Rule

- Two Equations–Two Variables
- Three Equations–Three Variables

Now let's see how determinants arise rather naturally in the process of solving systems of linear equations. We start by investigating two equations and two variables, and then extend our results to three equations and three variables.

• Two Equations–Two Variables

Instead of thinking of each system of linear equations in two variables as a different problem, let's see what happens when we attempt to solve the general system

$$a_{11}x + a_{12}y = k_1 \quad (1A)$$

$$a_{21}x + a_{22}y = k_2 \quad (1B)$$

once and for all, in terms of the unspecified real constants a_{11} , a_{12} , a_{21} , a_{22} , k_1 , and k_2 .

We proceed by multiplying equations (1A) and (1B) by suitable constants so that when the resulting equations are added, left side to left side and right side to right side, one of the variables drops out. Suppose we choose to eliminate y . What constant should we use to make the coefficients of y the same except for the signs? Multiply equation (1A) by a_{22} and (1B) by $-a_{12}$; then add:

$$\begin{array}{r} a_{22}(1A): \quad a_{11}a_{22}x + a_{12}a_{22}y = k_1a_{22} \\ -a_{12}(1B): \quad -a_{21}a_{12}x - a_{12}a_{22}y = -k_2a_{12} \\ \hline a_{11}a_{22}x - a_{21}a_{12}x + 0y = k_1a_{22} - k_2a_{12} \\ (a_{11}a_{22} - a_{21}a_{12})x = k_1a_{22} - k_2a_{12} \end{array}$$

$$x = \frac{k_1a_{22} - k_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \quad a_{11}a_{22} - a_{21}a_{12} \neq 0$$

What do the numerator and denominator remind you of? From your experience with determinants in the last two sections, you should recognize these expressions as

$$x = \frac{\begin{vmatrix} k_1 & a_{12} \\ k_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Similarly, starting with system (1A) and (1B) and eliminating x (this is left as an exercise), we obtain

$$y = \frac{\begin{vmatrix} a_{11} & k_1 \\ a_{21} & k_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

These results are summarized in Theorem 1, **Cramer's rule**, which is named after the Swiss mathematician G. Cramer (1704–1752).

Theorem 1 Cramer's Rule for Two Equations and Two Variables

Given the system

$$\begin{aligned} a_{11}x + a_{12}y &= k_1 \\ a_{21}x + a_{22}y &= k_2 \end{aligned} \quad \text{with} \quad D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

then

$$x = \frac{\begin{vmatrix} k_1 & a_{12} \\ k_2 & a_{22} \end{vmatrix}}{D} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_{11} & k_1 \\ a_{21} & k_2 \end{vmatrix}}{D}$$

The determinant D is called the **coefficient determinant**. If $D \neq 0$, then the system has exactly one solution, which is given by Cramer's rule. If, on the other hand, $D = 0$, then it can be shown that the system is either inconsistent and has no solutions or dependent and has an infinite number of solutions. We must use other methods, such as those discussed in Chapter 9, to determine the exact nature of the solutions when $D = 0$.

EXAMPLE 1 Solving a System with Cramer's Rule

Solve using Cramer's rule:
$$\begin{aligned} 3x - 5y &= 2 \\ -4x + 3y &= -1 \end{aligned}$$

Solution

$$\begin{aligned} D &= \begin{vmatrix} 3 & -5 \\ -4 & 3 \end{vmatrix} = -11 \\ x &= \frac{\begin{vmatrix} 2 & -5 \\ -1 & 3 \end{vmatrix}}{-11} = -\frac{1}{11} & y &= \frac{\begin{vmatrix} 3 & 2 \\ -4 & -1 \end{vmatrix}}{-11} = -\frac{5}{11} \end{aligned}$$

Matched Problem 1 Solve using Cramer's rule:
$$\begin{aligned} 3x + 2y &= -4 \\ -4x + 3y &= -10 \end{aligned}$$

EXPLORE-DISCUSS 1 Recall that a system of linear equations must have zero, one, or an infinite number of solutions. Discuss the number of solutions for the system

$$ax + 3y = b$$

$$4x + 2y = 8$$

where a and b are real numbers. Use Cramer's rule where appropriate and Gauss–Jordan elimination otherwise.

• Three Equations– Three Variables

Cramer's rule can be generalized completely for any size linear system that has the *same number of variables as equations*. However, it cannot be used to solve systems where the number of variables is not equal to the number of equations. In Theorem 2 we state without proof Cramer's rule for three equations and three variables.

Theorem 2

Cramer's Rule for Three Equations and Three Variables

Given the system

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= k_1 \\ a_{21}x + a_{22}y + a_{23}z &= k_2 \\ a_{31}x + a_{32}y + a_{33}z &= k_3 \end{aligned} \quad \text{with} \quad D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

then

$$x = \frac{\begin{vmatrix} k_1 & a_{12} & a_{13} \\ k_2 & a_{22} & a_{23} \\ k_3 & a_{32} & a_{33} \end{vmatrix}}{D} \quad y = \frac{\begin{vmatrix} a_{11} & k_1 & a_{13} \\ a_{21} & k_2 & a_{23} \\ a_{31} & k_3 & a_{33} \end{vmatrix}}{D} \quad z = \frac{\begin{vmatrix} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \\ a_{31} & a_{32} & k_3 \end{vmatrix}}{D}$$

You can easily remember these determinant formulas for x , y , and z if you observe the following:

1. Determinant D is formed from the coefficients of x , y , and z , keeping the same relative position in the determinant as found in the system of equations.
2. Determinant D appears in the denominators for x , y , and z .
3. The numerator for x can be obtained from D by replacing the coefficients of x (a_{11} , a_{21} , a_{31}) with the constants k_1 , k_2 , and k_3 , respectively. Similar statements can be made for the numerators for y and z .

EXAMPLE 2 Solving a System with Cramer's Rule

Solve using Cramer's rule:

$$\begin{aligned} x + y &= 2 \\ 3y - z &= -4 \\ x + z &= 3 \end{aligned}$$

Solution

$$D = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 2$$

$$x = \frac{\begin{vmatrix} 2 & 1 & 0 \\ -4 & 3 & -1 \\ 3 & 0 & 1 \end{vmatrix}}{2} = \frac{7}{2} \quad y = \frac{\begin{vmatrix} 1 & 2 & 0 \\ 0 & -4 & -1 \\ 1 & 3 & 1 \end{vmatrix}}{2} = -\frac{3}{2}$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & -4 \\ 1 & 0 & 3 \end{vmatrix}}{2} = -\frac{1}{2}$$

Matched Problem 2 Solve using Cramer's rule: $3x - z = 5$
 $x - y + z = 0$
 $x + y = 1$

In practice, Cramer's rule is rarely used to solve systems of order higher than 2 or 3 by hand, since more efficient methods are available utilizing computer methods. However, Cramer's rule is a valuable tool in more advanced theoretical and applied mathematics.

Answers to Matched Problems

1. $x = \frac{8}{17}, y = -\frac{46}{17}$ 2. $x = \frac{6}{5}, y = -\frac{1}{5}, z = -\frac{7}{5}$

EXERCISE 10-6

A

Solve Problems 1–8 using Cramer's rule.

1. $x + 2y = 1$
 $x + 3y = -1$

2. $x + 2y = 3$
 $x + 3y = 5$

3. $2x + y = 1$
 $5x + 3y = 2$

4. $x + 3y = 1$
 $2x + 8y = 0$

5. $2x - y = -3$
 $-x + 3y = 3$

6. $-3x + 2y = 1$
 $2x - 3y = -3$

7. $4x - 3y = 4$
 $3x + 2y = -2$

8. $5x + 2y = -1$
 $2x - 3y = 2$

B

Solve Problems 9–12 to 2 significant digits using Cramer's rule.

9. $0.9925x - 0.9659y = 0$
 $0.1219x + 0.2588y = 2,500$

10. $0.9877x - 0.9744y = 0$
 $0.1564x + 0.2250y = 1,900$

11. $0.9954x - 0.9942y = 0$
 $0.0958x + 0.1080y = 155$

12. $0.9973x - 0.9957y = 0$
 $0.0732x + 0.0924y = 112$

Solve Problems 13–20 using Cramer's rule:

- | | |
|---|---|
| <p>13. $x + y = 0$
$2y + z = -5$
$-x + z = -3$</p> <p>15. $x + y = 1$
$2y + z = 0$
$-y + z = 1$</p> <p>17. $3y + z = -1$
$x + 2z = 3$
$x - 3y = -2$</p> <p>19. $2y - z = -3$
$x - y - z = 2$
$x - y + 2z = 4$</p> | <p>14. $x + y = -4$
$2y + z = 0$
$-x + z = 5$</p> <p>16. $x + 3y = -3$
$2y + z = 3$
$-x + 3z = 7$</p> <p>18. $x - z = 3$
$2x - y = -3$
$x + y + z = 1$</p> <p>20. $2x + y = 2$
$x - y + z = -1$
$x + y + z = 2$</p> |
|---|---|

C

In Problems 21 and 22, use Cramer's rule to solve for x only.

- | | |
|---|---|
| <p>21. $2x - 3y + z = -3$
$-4x + 3y + 2z = -11$
$x - y - z = 3$</p> | <p>22. $x + 4y - 3z = 25$
$3x + y - z = 2$
$-4x + y + 2z = 1$</p> |
|---|---|

In Problems 23 and 24, use Cramer's rule to solve for y only.

- 23.** $12x - 14y + 11z = 5$
 $15x + 7y - 9z = -13$
 $5x - 3y + 2z = 0$
- 24.** $2x - y + 4z = 15$
 $-x + y + 2z = 5$
 $3x + 4y - 2z = 4$

In Problems 25 and 26, use Cramer's rule to solve for z only.

- 25.** $3x - 4y + 5z = 18$
 $-9x + 8y + 7z = -13$
 $5x - 7y + 10z = 33$
- 26.** $13x + 11y + 10z = 2$
 $10x + 8y + 7z = 1$
 $8x + 5y + 4z = 4$

It is clear that $x = 0, y = 0, z = 0$ is a solution to each of the systems given in Problems 27 and 28. Use Cramer's rule to determine whether this solution is unique. [Hint: If $D \neq 0$, what can you conclude? If $D = 0$, what can you conclude?]

- | | |
|---|--|
| <p>27. $x - 4y + 9z = 0$
$4x - y + 6z = 0$
$x - y + 3z = 0$</p> | <p>28. $3x - y + 3z = 0$
$5x + 5y - 9z = 0$
$-2x + y - 3z = 0$</p> |
|---|--|

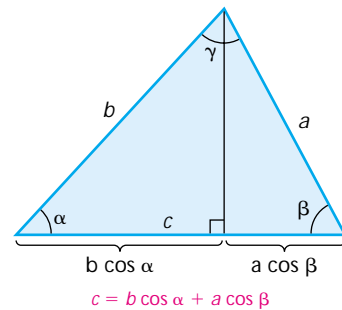
- 29.** Prove Theorem 1 for y .

- 30.** (Omit this problem if you have not studied trigonometry.) The angles $\alpha, \beta,$ and γ and the sides $a, b,$ and c of a triangle (see the figure) satisfy

$$\begin{aligned} c &= b \cos \alpha + a \cos \beta \\ b &= c \cos \alpha + a \cos \gamma \\ a &= c \cos \beta + b \cos \gamma \end{aligned}$$

Use Cramer's rule to express $\cos \alpha$ in terms of $a, b,$ and c , thereby deriving the familiar law of cosines from trigonometry:

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}$$



APPLICATIONS



- 31. Revenue Analysis.** A supermarket sells two brands of coffee: brand A at $\$p$ per pound and brand B at $\$q$ per pound. The daily demand equations for brands A and B are, respectively,

$$\begin{aligned} x &= 200 - 6p + 4q \\ y &= 300 + 2p - 3q \end{aligned} \quad (1)$$

(both in pounds). The daily revenue R is given by

$$R = xp + yq$$

- (A) To analyze the effect of price changes on the daily revenue, an economist wants to express the daily revenue R in terms of p and q only. Use (1) to eliminate x and y in the equation for R , thus expressing the daily revenue in terms of p and q .
- (B) To analyze the effect of changes in demand on the daily revenue, the economist now wants to express the daily revenue in terms of x and y only. Use Cramer's rule to solve system (1) for p and q in terms of x and y and then express the daily revenue R in terms of x and y .

32. Revenue Analysis. A company manufactures ten-speed and three-speed bicycles. The weekly demand equations are

$$\begin{aligned} p &= 230 - 10x + 5y \\ q &= 130 + 4x - 4y \end{aligned} \quad (2)$$

where $\$p$ is the price of a ten-speed bicycle, $\$q$ is the price of a three-speed bicycle, x is the weekly demand for ten-

speed bicycles, and y is the weekly demand for three-speed bicycles. The weekly revenue R is given by

$$R = xp + yq$$

- (A) Use (2) to express the daily revenue in terms of x and y only.
 (B) Use Cramer's rule to solve system (2) for x and y in terms of p and q , and then express the daily revenue R in terms of p and q only.

CHAPTER 10 GROUP ACTIVITY Using Matrices to Find Cost, Revenue, and Profit

A toy distributor purchases model train components from various suppliers and packages these components in three different ready-to-run train sets: the Limited, the Empire, and the Comet. The components used in each set are listed in Table 1. For convenience, the total labor time (in minutes) required to prepare a set for shipping is included as a component.

TABLE 1 Product Components

Components	Train Sets		
	Limited	Empire	Comet
Locomotives	1	1	2
Cars	5	6	8
Track pieces	20	24	32
Track switches	1	2	4
Power pack	1	1	1
Labor (min)	15	18	24

The current costs of the components are given in Table 2, and the distributor's selling prices for the sets are given in Table 3.

TABLE 2 Component Costs

Components	Cost per unit
Locomotive	\$12.52
Car	\$1.43
Track piece	\$0.25
Track switch	\$2.29
Power pack	\$12.54
Labor (per min)	\$0.15

TABLE 3 Selling Prices

Set	Price
Limited	\$54.60
Empire	\$62.28
Comet	\$81.15

The distributor has just received the order shown in Table 4 from a retail toy store.

TABLE 4 Customer Order

Set	Quantity
Limited	48
Empire	24
Comet	12

The distributor wants to store the information in each table in a matrix and use matrix operations to find the following information:

- The inventory (parts and labor) required to fill the order
 - The cost (parts and labor) of filling the order
 - The revenue (sales) received from the customer
 - The profit realized on the order
- (A) Use a single letter to designate the matrix representing each table, and write matrix expressions in terms of these letters that will provide the required information. Discuss the size of the matrix you must use to represent each table so that all the pertinent matrix operations are defined.
- (B) Evaluate the matrix expressions in part A.

Shortly after filling the order in Table 4, a supplier informs the distributor that the cars and locomotives used in these train sets are no longer available. The distributor currently has 30 locomotives and 134 cars in stock.

- (C) How many train sets of each type can the distributor produce using all the available locomotives and cars? Assume that the distributor has unlimited quantities of the other components used in these sets.
- (D) How much profit will the distributor make if all these sets are sold? If there is more than one way to use all the available locomotives and cars, which one will produce the largest profit?

Chapter 10 Review

10-1 MATRICES: BASIC OPERATIONS

Two matrices are **equal** if they are the same size and their corresponding elements are equal. The **sum of two matrices** of the same size is a matrix with elements that are the sums of the corresponding elements of the two given matrices. Matrix addition is **commutative** and **associative**. A matrix with all zero elements is called the **zero matrix**. The **negative of a matrix M** , denoted $-M$, is a matrix with elements that are the negatives of the elements in M . If A and B are matrices of the same size, then we define **subtraction** as follows: $A - B = A + (-B)$. The **product of a number k and a matrix M** , denoted by kM , is a matrix formed by multiplying each element of M by k . The

product of a $1 \times n$ row matrix and an $n \times 1$ column matrix is a 1×1 matrix given by

$$\begin{array}{c}
 1 \times n \\
 [a_1 \quad a_2 \quad \cdots \quad a_n]
 \end{array}
 \begin{array}{c}
 n \times 1 \\
 \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
 \end{array}
 =
 \begin{array}{c}
 1 \times 1 \\
 [a_1b_1 + a_2b_2 + \cdots + a_nb_n]
 \end{array}$$

If A is an $m \times p$ matrix and B is a $p \times n$ matrix, then the **matrix**