


★★ 37. **Geometry.** The graph of  $f(x) = ax^2 + bx + c$  passes through the points  $(1, k_1)$ ,  $(2, k_2)$ , and  $(3, k_3)$ . Determine  $a$ ,  $b$ , and  $c$  for:

- (A)  $k_1 = -2, k_2 = 1, k_3 = 6$   
 (B)  $k_1 = 4, k_2 = 3, k_3 = -2$   
 (C)  $k_1 = 8, k_2 = -5, k_3 = 4$

★★ 38. **Geometry.** Repeat Problem 37 if the graph passes through the points  $(-1, k_1)$ ,  $(0, k_2)$ , and  $(1, k_3)$ .

 Check your answers in Problems 37 and 38 by graphing  $y = f(x)$  on a graphing utility and verifying that the graph passes through the indicated points.

39. **Diets.** A biologist has available two commercial food mixes with the following percentages of protein and fat:

Mix	Protein (%)	Fat (%)
A	20	2
B	10	6

How many ounces of each mix should be used to prepare each of the diets listed in the following table?

	Diet		
	1	2	3
Protein	20 oz	10 oz	10 oz
Fat	6 oz	4 oz	6 oz

## SECTION 10-4 Determinants

- Determinants
- Second-Order Determinants
- Third-Order Determinants
- Higher-Order Determinants

### • Determinants

In this section we are going to associate with each square matrix a real number, called the **determinant** of the matrix. If  $A$  is a square matrix, then the determinant of  $A$  is denoted by **det A**, or simply by writing the array of elements in  $A$  using vertical lines in place of square brackets. For example,

$$\det \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix} = \begin{vmatrix} 2 & -3 \\ 5 & 1 \end{vmatrix} \quad \det \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -7 \\ -2 & 1 & 6 \end{bmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 5 & -7 \\ -2 & 1 & 6 \end{vmatrix}$$

A determinant of **order  $n$**  is a determinant with  $n$  rows and  $n$  columns. In this section we concentrate most of our attention on determining the values of determinants of orders 2 and 3. But many of the results and procedures discussed can be generalized completely to determinants of order  $n$ .

### • Second-Order Determinants

In general, a **second-order determinant** is written as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

and represents a real number as given in Definition 1.

## DEFINITION 1

## Value of a Second-Order Determinant

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \quad (1)$$

Formula (1) is easily remembered if you notice that the expression on the right is the product of the **principal diagonal**, from upper left to lower right, minus the product of the **secondary diagonal**, from lower left to upper right.

## EXAMPLE 1 Evaluating a Second-Order Determinant

$$\begin{vmatrix} -1 & 2 \\ -3 & -4 \end{vmatrix} = (-1)(-4) - (-3)(2) = 4 - (-6) = 10$$

## Matched Problem 1

Find:  $\begin{vmatrix} 3 & -5 \\ 4 & -2 \end{vmatrix}$

## • Third-Order Determinants

A determinant of order 3 is a square array of nine elements and represents a real number given by Definition 2, which is a special case of the general definition of the value of an  $n$ th-order determinant. Note that each term in the expansion on the right of equation (2) contains exactly one element from each row and each column.

## DEFINITION 2

## Value of a Third-Order Determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{21}a_{32}a_{13} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} \quad (2)$$

Don't panic! You don't need to memorize formula (2). After we introduce the ideas of *minor* and *cofactor* below, we will state a theorem that can be used to obtain the same result with much less trouble.

The **minor of an element** in a third-order determinant is a second-order determinant obtained by deleting the row and column that contains the element. For example, in the determinant in formula (2),

$$\text{Minor of } a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \quad \begin{array}{|c|c|c|} \hline a_{11} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \\ \hline \end{array} \quad \text{Deletions are usually done mentally.}$$

$$\text{Minor of } a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \quad \begin{array}{|c|c|c|} \hline a_{11} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \\ \hline \end{array}$$

EXPLORE-DISCUSS 1 Write the minors of the other seven elements in the determinant in formula (2).

A quantity closely associated with the minor of an element is the **cofactor of an element**  $a_{ij}$  (from the  $i$ th row and  $j$ th column), which is the product of the minor of  $a_{ij}$  and  $(-1)^{i+j}$ .

## DEFINITION 3

**Cofactor**

$$\text{Cofactor of } a_{ij} = (-1)^{i+j}(\text{Minor of } a_{ij})$$

Thus, a cofactor of an element is nothing more than a signed minor. The sign is determined by raising  $-1$  to a power that is the sum of the numbers indicating the row and column in which the element appears. Note that  $(-1)^{i+j}$  is 1 if  $i + j$  is even and  $-1$  if  $i + j$  is odd. Thus, if we are given the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

then

$$\text{Cofactor of } a_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\text{Cofactor of } a_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

**EXAMPLE 2** Finding Cofactors

Find the cofactors of  $-2$  and  $5$  in the determinant

$$\begin{vmatrix} -2 & 0 & 3 \\ 1 & -6 & 5 \\ -1 & 2 & 0 \end{vmatrix}$$

**Solution**

$$\begin{aligned} \text{Cofactor of } -2 &= (-1)^{1+1} \begin{vmatrix} -6 & 5 \\ 2 & 0 \end{vmatrix} = \begin{vmatrix} -6 & 5 \\ 2 & 0 \end{vmatrix} \\ &= (-6)(0) - (2)(5) = -10 \end{aligned}$$

$$\begin{aligned} \text{Cofactor of } 5 &= (-1)^{2+3} \begin{vmatrix} -2 & 0 \\ -1 & 2 \end{vmatrix} = - \begin{vmatrix} -2 & 0 \\ -1 & 2 \end{vmatrix} \\ &= -[(-2)(2) - (-1)(0)] = 4 \end{aligned}$$

**Matched Problem 2** Find the cofactors of 2 and 3 in the determinant in Example 2.

[*Note:* The sign in front of the minor,  $(-1)^{i+j}$ , can be determined rather mechanically by using a checkerboard pattern of + and - signs over the determinant, starting with + in the upper left-hand corner:

$$\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array}$$

Use either the checkerboard or the exponent method—whichever is easier for you—to determine the sign in front of the minor.]

Now we are ready for the key theorem of this section, Theorem 1. This theorem provides us with an efficient step-by-step procedure, called an algorithm, for evaluating third-order determinants.

**Theorem 1 Value of a Third-Order Determinant**

The value of a determinant of order 3 is the sum of three products obtained by multiplying each element of any one row (or each element of any one column) by its cofactor.

To prove this theorem we must show that the expansions indicated by the theorem for any row or any column (six cases) produce the expression on the right of formula (2). Proofs of special cases of this theorem are left to the C problems in Exercise 10-4.

**EXAMPLE 3 Evaluating a Third-Order Determinant**

Evaluate

$$\begin{vmatrix} 2 & -2 & 0 \\ -3 & 1 & 2 \\ 1 & -3 & -1 \end{vmatrix}$$

by expanding by:

- (A) The first row      (B) The second column

**Solutions**

$$\begin{aligned}
 \text{(A)} \quad \begin{vmatrix} 2 & -2 & 0 \\ -3 & 1 & 2 \\ 1 & -3 & -1 \end{vmatrix} &= a_{11} \begin{pmatrix} \text{Cofactor} \\ \text{of } a_{11} \end{pmatrix} + a_{12} \begin{pmatrix} \text{Cofactor} \\ \text{of } a_{12} \end{pmatrix} + a_{13} \begin{pmatrix} \text{Cofactor} \\ \text{of } a_{13} \end{pmatrix} \\
 &= 2 \left( (-1)^{1+1} \begin{vmatrix} 1 & 2 \\ -3 & -1 \end{vmatrix} \right) + (-2) \left( (-1)^{1+2} \begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix} \right) + 0 \\
 &= (2)(1)[(1)(-1) - (-3)(2)] + (-2)(-1)[(-3)(-1) - (1)(2)] \\
 &= (2)(5) + (2)(1) = 12
 \end{aligned}$$

$$\begin{aligned}
 \text{(B)} \quad \begin{vmatrix} 2 & -2 & 0 \\ -3 & 1 & 2 \\ 1 & -3 & -1 \end{vmatrix} &= a_{12} \begin{pmatrix} \text{Cofactor} \\ \text{of } a_{12} \end{pmatrix} + a_{22} \begin{pmatrix} \text{Cofactor} \\ \text{of } a_{22} \end{pmatrix} + a_{32} \begin{pmatrix} \text{Cofactor} \\ \text{of } a_{32} \end{pmatrix} \\
 &= (-2) \left( (-1)^{1+2} \begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix} \right) + (1) \left( (-1)^{2+2} \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} \right) \\
 &\quad + (-3) \left( (-1)^{3+2} \begin{vmatrix} 2 & 0 \\ -3 & 2 \end{vmatrix} \right) \\
 &= (-2)(-1)[(-3)(-1) - (1)(2)] + (1)(1)[(2)(-1) - (1)(0)] \\
 &\quad + (-3)(-1)[(2)(2) - (-3)(0)] \\
 &= (2)(1) + (1)(-2) + (3)(4) = 12
 \end{aligned}$$

### Matched Problem 3 Evaluate

$$\begin{vmatrix} 2 & 1 & -1 \\ -2 & -3 & 0 \\ -1 & 2 & 1 \end{vmatrix}$$

by expanding by:

- (A) The first row      (B) The third column

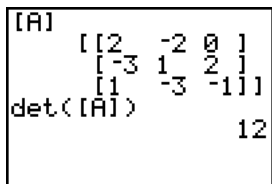


FIGURE 1



Most graphing utilities will evaluate determinants. Figure 1 shows the evaluation of the determinant in Example 3.

### • Higher-Order Determinants

Theorem 1 and the definitions of minor and cofactor generalize completely for determinants of order higher than 3. These concepts are illustrated for a fourth-order determinant in the next example.

### EXAMPLE 4 Evaluating a Fourth-Order Determinant

Given the fourth-order determinant

$$\begin{vmatrix} 0 & -1 & 0 & 2 \\ -5 & -6 & 0 & -3 \\ 4 & 5 & -2 & 6 \\ 0 & 3 & 0 & -4 \end{vmatrix}$$

- (A) Find the minor in determinant form of the element 3.  
 (B) Find the cofactor in determinant form of the element  $-5$ .  
 (C) Find the value of the fourth-order determinant.

**Solutions** (A) Minor of 3 =  $\begin{vmatrix} 0 & 0 & 2 \\ -5 & 0 & -3 \\ 4 & -2 & 6 \end{vmatrix}$

(B) Cofactor of  $-5 = (-1)^{2+1} \begin{vmatrix} -1 & 0 & 2 \\ 5 & -2 & 6 \\ 3 & 0 & -4 \end{vmatrix} = - \begin{vmatrix} -1 & 0 & 2 \\ 5 & -2 & 6 \\ 3 & 0 & -4 \end{vmatrix}$

- (C) Generalizing Theorem 1, the value of this fourth-order determinant is the sum of four products obtained by multiplying each element of any one row (or each element of any one column) by its cofactor. The work involved in this evaluation is greatly reduced if we choose the row or column with the greatest number of 0's. Since column 3 has three 0's, we expand along this column:

$$\begin{aligned} \begin{vmatrix} 0 & -1 & 0 & 2 \\ -5 & -6 & 0 & -3 \\ 4 & 5 & -2 & 6 \\ 0 & 3 & 0 & -4 \end{vmatrix} &= 0 + 0 + (-2)(-1)^{3+3} \begin{vmatrix} 0 & -1 & 2 \\ -5 & -6 & -3 \\ 0 & 3 & -4 \end{vmatrix} + 0 \\ &= (-2) \begin{vmatrix} 0 & -1 & 2 \\ -5 & -6 & -3 \\ 0 & 3 & -4 \end{vmatrix} \quad \text{Expand this determinant} \\ &= (-2) \left( 0 + (-5)(-1)^{2+1} \begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix} + 0 \right) \\ &= (-2)(-5)(-1)(-2) = 20 \end{aligned}$$

**Matched Problem 4** Repeat Example 4 for the following fourth-order determinant:

$$\begin{vmatrix} 0 & 4 & -2 & 0 \\ -3 & 3 & -1 & 2 \\ 0 & 6 & 0 & 0 \\ 5 & -6 & -5 & -4 \end{vmatrix}$$

**EXPLORE-DISCUSS 2** Write a checkerboard pattern of + and - signs for a fourth-order determinant, and use it to determine the signs of the minors in Example 4.

**Remark.** Where are determinants used? Many equations and formulas have particularly simple and compact representations in determinant form that are easily remembered. (See Problems 50–54 in Exercise 10-5). Also, in Section 10-6 we will see that the solutions to certain systems of equations can be expressed in terms of determinants. In addition, determinants are involved in theoretical work in advanced mathematics courses. For example, it can be shown that the inverse of a square matrix exists if and only if its determinant is not 0.

#### Answers to Matched Problems

1. 14      2. Cofactor of 2 = 13; cofactor of 3 = -4      3. (A) 3      (B) 3  
 4. (A)  $\begin{vmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 5 & -5 & -4 \end{vmatrix}$       (B)  $-\begin{vmatrix} 0 & 4 & 0 \\ -3 & 3 & 2 \\ 0 & 6 & 0 \end{vmatrix}$       (C) -24

## EXERCISE 10-4

### A

Evaluate each second-order determinant in Problems 1–6.

1.  $\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix}$       2.  $\begin{vmatrix} 8 & -3 \\ 4 & 1 \end{vmatrix}$   
 3.  $\begin{vmatrix} 3 & -7 \\ -5 & 6 \end{vmatrix}$       4.  $\begin{vmatrix} 9 & -2 \\ 4 & 0 \end{vmatrix}$   
 5.  $\begin{vmatrix} 4.3 & -1.2 \\ -5.1 & 3.7 \end{vmatrix}$       6.  $\begin{vmatrix} -0.7 & -2.3 \\ 1.9 & -4.8 \end{vmatrix}$

Problems 7–14 pertain to the determinant below:

$$\begin{vmatrix} 5 & -1 & -3 \\ 3 & 4 & 6 \\ 0 & -2 & 8 \end{vmatrix}$$


Write the minor of each element given in Problems 7–10. Leave the answer in determinant form.

7.  $a_{11}$       8.  $a_{33}$   
 9.  $a_{23}$       10.  $a_{12}$

Write the cofactor of each element given in Problems 11–14, and evaluate each.

11.  $a_{11}$       12.  $a_{33}$   
 13.  $a_{23}$       14.  $a_{12}$

Evaluate Problems 15–20 using cofactors.

 Check your answers to Problems 15–20 on a graphing utility.

15.  $\begin{vmatrix} 1 & 0 & 0 \\ -2 & 4 & 3 \\ 5 & -2 & 1 \end{vmatrix}$       16.  $\begin{vmatrix} 2 & -3 & 5 \\ 0 & -3 & 1 \\ 0 & 6 & 2 \end{vmatrix}$

17.  $\begin{vmatrix} 0 & 1 & 5 \\ 3 & -7 & 6 \\ 0 & -2 & -3 \end{vmatrix}$       18.  $\begin{vmatrix} 4 & -2 & 0 \\ 9 & 5 & 4 \\ 1 & 2 & 0 \end{vmatrix}$   
 19.  $\begin{vmatrix} -1 & 2 & -3 \\ -2 & 0 & -6 \\ 4 & -3 & 2 \end{vmatrix}$       20.  $\begin{vmatrix} 0 & 2 & -1 \\ -6 & 3 & 1 \\ 7 & -9 & -2 \end{vmatrix}$

### B


Given the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

write the cofactor in determinant form of each element in Problems 21–24.

21.  $a_{11}$       22.  $a_{44}$   
 23.  $a_{43}$       24.  $a_{23}$

Evaluate each determinant in Problems 25–34 using cofactors.

 Check your answers to Problems 25–34 on a graphing utility.

25.  $\begin{vmatrix} 3 & -2 & -8 \\ -2 & 0 & -3 \\ 1 & 0 & -4 \end{vmatrix}$       26.  $\begin{vmatrix} 4 & -4 & 6 \\ 2 & 8 & -3 \\ 0 & -5 & 0 \end{vmatrix}$   
 27.  $\begin{vmatrix} 1 & 4 & 1 \\ 1 & 1 & -2 \\ 2 & 1 & -1 \end{vmatrix}$       28.  $\begin{vmatrix} 3 & 2 & 1 \\ -1 & 5 & 1 \\ 2 & 3 & 1 \end{vmatrix}$

$$29. \begin{vmatrix} 1 & 4 & 3 \\ 2 & 1 & 6 \\ 3 & -2 & 9 \end{vmatrix} \qquad 30. \begin{vmatrix} 4 & -6 & 3 \\ -1 & 4 & 1 \\ 5 & -6 & 3 \end{vmatrix}$$

$$31. \begin{vmatrix} 2 & 6 & 1 & 7 \\ 0 & 3 & 0 & 0 \\ 3 & 4 & 2 & 5 \\ 0 & 9 & 0 & 2 \end{vmatrix} \qquad 32. \begin{vmatrix} 0 & 1 & 0 & 1 \\ 2 & 4 & 7 & 6 \\ 0 & 3 & 0 & 1 \\ 0 & 6 & 2 & 5 \end{vmatrix}$$

$$33. \begin{vmatrix} -2 & 0 & 0 & 0 & 0 \\ 9 & -1 & 0 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ -1 & 4 & 2 & 2 & 0 \\ 7 & -2 & 3 & 5 & 5 \end{vmatrix}$$

$$34. \begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix}$$

If  $A$  is a  $3 \times 3$  matrix,  $\det A$  can be evaluated by the following **diagonal expansion**. Form a  $3 \times 5$  matrix by augmenting  $A$  on the right with its first two columns, and compute the diagonal products  $p_1, p_2, \dots, p_6$  indicated by the arrows:

$$\begin{array}{cccccc} & \begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix} & \text{Diagonal expansion formula} \\ p_4 & p_5 & p_6 & p_1 & p_2 & p_3 \end{array}$$

The determinant of  $A$  is given by [compare with formula (2)]

$$\begin{aligned} \det A &= p_1 + p_2 + p_3 - p_4 - p_5 - p_6 \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\ &\quad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \end{aligned}$$

[Caution: The diagonal expansion procedure works only for  $3 \times 3$  matrices. Do not apply it to matrices of any other size.]

Use the diagonal expansion formula to evaluate the determinants in Problems 35 and 36.

$$35. \begin{vmatrix} 2 & 6 & -1 \\ 5 & 3 & -7 \\ -4 & -2 & 1 \end{vmatrix} \qquad 36. \begin{vmatrix} 4 & 1 & -5 \\ 1 & 2 & -6 \\ -3 & -1 & 7 \end{vmatrix}$$

A square matrix is called an **upper triangular matrix** if all elements below the principal diagonal are zero. In Problems 37–40, determine whether the statement is true or false. If true, explain why. If false, give a counterexample.

37. If the determinant of an upper triangular matrix is 0, then the elements on the principal diagonal are all 0.

38. If  $A$  and  $B$  are upper triangular matrices, then  $\det(A + B) = \det A + \det B$ .

39. The determinant of an upper triangular matrix is the product of the elements on the principal diagonal.

40. If  $A$  and  $B$  are upper triangular matrices, then  $\det(AB) = (\det A)(\det B)$ .

## C

In Problems 41–46, all the letters represent real numbers. Find an equation that each pair of determinants satisfies, and describe the relationship between the two determinants verbally.

$$41. \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \begin{vmatrix} c & d \\ a & b \end{vmatrix} \qquad 42. \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \begin{vmatrix} b & a \\ d & c \end{vmatrix}$$

$$43. \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \begin{vmatrix} ka & b \\ kc & d \end{vmatrix} \qquad 44. \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \begin{vmatrix} a & b \\ kc & kd \end{vmatrix}$$

$$45. \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \begin{vmatrix} kc + a & kd + b \\ c & d \end{vmatrix}$$

$$46. \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \begin{vmatrix} a & ka + b \\ c & kc + d \end{vmatrix}$$

47. Show that the expansion of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

by the first column is the same as its expansion by the third row.

48. Repeat Problem 47, using the second row and the third column.

49. If

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$$

show that  $\det(AB) = \det A \cdot \det B$ .

50. If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

show that  $\det(AB) = \det A \cdot \det B$ .

If  $A$  is an  $n \times n$  matrix and  $I$  is the  $n \times n$  identity matrix, then the function  $f(x) = |xI - A|$  is called the **characteristic polynomial** of  $A$ , and the zeros of  $f(x)$  are called the **eigenvalues** of  $A$ . Characteristic polynomials and eigenvalues have many important applications that are discussed in more

advanced treatments of matrices. In Problems 51–54, find the characteristic polynomial and the eigenvalues of each matrix.

$$51. \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$$

$$52. \begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix}$$

$$53. \begin{bmatrix} 4 & -4 & 0 \\ 2 & -2 & 0 \\ 4 & -8 & -4 \end{bmatrix}$$

$$54. \begin{bmatrix} -2 & 2 & 0 \\ -1 & 1 & 0 \\ -2 & 4 & 2 \end{bmatrix}$$

## SECTION 10-5 Properties of Determinants

- Discussion of Determinant Properties
- Summary of Determinant Properties

Determinants have a number of useful properties that can greatly reduce the labor in evaluating determinants of order 3 or greater. These properties and their use are the subject matter for this section.

- Discussion of Determinant Properties

We now state and discuss five general determinant properties in the form of theorems. Because the proofs for the general cases of these theorems are involved and notationally difficult, we will sketch only informal proofs for determinants of order 3. The theorems, however, apply to determinants of any order.

### Theorem 1

#### Multiplying a Row or Column by a Constant

If each element of any row (or column) of a determinant is multiplied by a constant  $k$ , the new determinant is  $k$  times the original.

**Partial Proof** Let  $C_{ij}$  be the cofactor of  $a_{ij}$ . Then expanding by the first row, we have

$$\begin{aligned} \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= ka_{11}C_{11} + ka_{12}C_{12} + ka_{13}C_{13} \\ &= k(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}) \\ &= k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{aligned}$$

Theorem 1 also states that a factor common to all elements of a row (or column) can be taken out as a factor of the determinant.