Stat 762

The Accept-Reject Algorithm

The basic idea of the Rejection (or Accept-Reject) Algorithm is the following.

Let $X_1, X_2, \ldots$ be i.i.d. (possibly multivariate), and let A be a (measurable) set in their range. (A is our acceptance region.)

Theorem 1. *Let $N = \min\{n : X_n \in A\}$. Then*

1. *$N \sim$ geometric(p), where $p = \mathrm{P}(X_1 \in A)$, and*
2. *for any (measurable) C in the range of X_i,*

$$\mathrm{P}(X_N \in C) = \frac{\mathrm{P}(X_1 \in A \cap C)}{\mathrm{P}(X_1 \in A)}.$$

Note that X_N has a random index.

Proof. The first part is clear. For the second, $\mathrm{P}(X_N \in C) = \sum_{n=1}^{\infty} \mathrm{P}(X_N \in C,\ N = n)$. But

$$\begin{aligned}\{X_N \in C,\ N = n\} &= \{X_n \in C,\quad N = n\}\\ &= \{X_n \in C,\quad X_n \in A,\ X_j \in A^c \text{ for } j \leq n-1\}\\ &= \{X_n \in A \cap C,\quad X_j \in A^c \text{ for } j \leq n-1\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{P}(X_N \in C,\ N = n) &= \mathrm{P}(X_n \in A \cap C)\,\mathrm{P}(X_j \in A^c \text{ for } j \leq n-1)\\ &= \mathrm{P}(X_1 \in A \cap C)\,\mathrm{P}(X_j \in A^c \text{ for } j \leq n-1).\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{P}(X_N \in C) &= \mathrm{P}(X_1 \in A \cap C) \sum_n \mathrm{P}(X_j \in A^c \text{ for } j \leq n-1)\\ &= \mathrm{P}(X_1 \in A \cap C) \sum_n q^{n-1} \qquad (q = 1-p)\\ &= \mathrm{P}(X_1 \in A \cap C)\,\frac{1}{1-q} = \frac{\mathrm{P}(X_1 \in A \cap C)}{\mathrm{P}(X_1 \in A)}.\end{aligned}$$

□