Assignments for Chapter 7

Section	Problems
7.2.1-2	9, A29, A30
7.2.3	22(a,c*), 25(b)*
Equivariance	6.42, 35, A31
7.3.1-2	38, 39, 66(a-c)
7.3.3	55(a), 57, 66(d)
7.3.4	62

Additional Problems:

A29: Find the MLE of n based on a single observation X from the binomial(n, p) distribution, p known. Apply this for n tosses of a fair coin where $X = 37$; where $X = 0$.

A30: Show that the MLE of (α, β) based on a sample $X_1, \dots, X_n$ from the gamma(α, β) distribution is given implicitly by the equations $\hat{\alpha}\hat{\beta} = \overline{X}$ and $\psi(\hat{\alpha}) + \ln(\hat{\beta}) = \ln(G)$ where $\psi(x) = d \ln\Gamma(x)/dx$ (the digamma function) and G is the geometric mean of the sample. (See ECR14.)

A31: Suppose X is a random variable or vector taking values in $\mathcal{X}$ with distribution indexed by $\theta \in \Theta$. Suppose $\mathcal{G}$ is a group of transformations of $\mathcal{X}$ and that the family of distributions of X is invariant under $\mathcal{G}$. Each $g \in \mathcal{G}$ induces a map $\overline{g}$ on Θ by the relation $\mathrm{P}_\theta(g(\mathsf{X}) \in B) = \mathrm{P}_{\overline{g}(\theta)}(X \in B)$, as discussed in class. Let $\overline{\mathcal{G}}$ be the set of induced maps $\overline{g}$.

a. Show that $\overline{\mathcal{G}}$ is a group.

b. Let $\phi : \mathcal{G} \to \overline{\mathcal{G}}$ be defined by $\phi(g) = \overline{g}$. Show that ϕ a group homomorphism.

c. Show that it is possible to have $\overline{g}_1 = \overline{g}_2$ but $g_1 \neq g_2$.

*Notes:

Problem 7.22(c) Find the posterior distribution of θ *without finding or using the density* m *of part (b) at all.* To do this, do (a), then rewrite the combined exponent as a quadratic in θ in such a way that the "kernel" of the pdf looks like that of a normal pdf. (The density m is found in part (b), which I am not assigning.)

Note also that equation(7.2.10) assumes $n = 1$. You will end up with the appropriate modification for $n > 1$.

Problem 7.25(b) We should really write the first assumption in the form

$$X_i|(\theta_1, \dots, \theta_n) \sim f(x|\theta_i), \quad i = 1, \dots, n, \quad \text{independent},$$

as the independence is conditional on the whole sample of θ's.

Problem 7.35(a) Show equivariant, not invariant.

Extra Credit:

ECR14 Show that you actually get a maximum in problem A30.

ECR15: Let X take values in $\mathcal{X}$, and let $\mathcal{G}$ be a group acting on $\mathcal{X}$. Let the family $\{P_\theta; \mu \in \Theta\}$ be invariant under $\mathcal{G}$, and assume moreover that $\overline{\mathcal{G}}$ acts transitively on Θ. Let $T(x)$ be a $\mathcal{G}$-invariant function of x. Show that $T(X)$ is ancillary for μ.

Definition: $\overline{\mathcal{G}}$ acts transitively on Θ if for any two elements μ and μ' in Θ there exists a $\overline{g} \in \overline{\mathcal{G}}$ such that $\overline{g}(\mu) = \mu'$.

ECR16: Prove or disprove: a location-scale family (on the real line) is complete.