Casella and Berger, 2nd ed., Problem 1.45

I intended to have you do this problem for the probability function P_X defined in equation (1.4.2):

$$\mathrm{P}_X(A) = \mathrm{P}(X \in A).$$

For this problem, a good way to express this is

$$\mathrm{P}_X(A) = \mathrm{P}(X^{-1}(A)). \tag{1}$$

Here $A \in \mathcal{B}$, where $\mathcal{B}$ is a σ-algebra of subsets of $\mathbb{R}$. In particular, A is a subset of $\mathbb{R}$. It's not really necessary to know what sets $\mathcal{B}$ contains in order to do this problem.

You need to show the Kolmogorov axioms for P_X. These are:

a. $\mathrm{P}_X(A) \geq 0$ for any A.

b. $\mathrm{P}_X(\mathbb{R}) = 1$.

c. If $A_1, A_2, \ldots \in \mathcal{B}$ are pairwise disjoint, then $\mathrm{P}_X(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathrm{P}_X(A_i)$.

You will need to use equation (1), rules about inverse images, and the fact that P satisfies the Kolmogorov axioms.

A note on notation: The text consistently writes $\mathrm{P}_X(X \in A)$. This is poor notation. We should write either $\mathrm{P}_X(A)$ or $\mathrm{P}(X \in A)$.