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Short Course on Multivariate Analysis – Part I

We write all vectors as columns; $'$ denotes transpose. The dot product of the vectors $\mathbf{a}$ and $\mathbf{b}$ is thus the matrix product $\mathbf{a}'\mathbf{b}$.

Let $\mathbf{X} = [X_1, \ldots, X_n]'$, where the components have a joint distribution. We call $\mathbf{X}$ a *random vector*. Similarly, if the mn components of the $m \times n$ matrix $\mathbf{M} = [M_{ij}]$ have a joint distribution, then we call $\mathbf{M}$ a *random matrix*.

We define the *mean* of a random vector or matrix componentwise: $\mathrm{E}(\mathbf{X}) = [\mathrm{E}(X_1), \ldots, \mathrm{E}(X_n)]'$ and $\mathrm{E}(\mathbf{M}) = [\mathrm{E}(M_{ij})]$. Notation: $\mathrm{E}(\mathbf{X}) = \boldsymbol{\mu}$.

Example: If $\mathbf{X} = [X_1, X_2]'$ and $\mathbf{a} = [a_1, a_2]'$, then $\mathbf{a}'\mathbf{X} = a_1X_1 + a_2X_2$ and $\mathbf{X}'\mathbf{X} = X_1^2 + X_2^2$. The matrix

$$(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})' = \begin{bmatrix} (X_1 - \mu_1)^2 & (X_1 - \mu_1)(X_2 - \mu_2) \\ (X_1 - \mu_1)(X_2 - \mu_2) & (X_2 - \mu_2)^2 \end{bmatrix}$$

is symmetric for all observations. We have

$$\mathrm{E}(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})' = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}.$$

In general, the *covariance matrix* of the vector $\mathbf{X}$ is the symmetric matrix

$$\boldsymbol{\Sigma} = \mathrm{E}(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})' = [\sigma_{ij}].$$

Its diagonal elements are variances, and its off-diagonal elements, covariances. We will subscript $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ by $\mathbf{X}$, ..., if necessary.

1. $\mathrm{E}(\mathbf{a}'\mathbf{X}) = \mathbf{a}'\mathrm{E}(\mathbf{X})$.

2. $\mathrm{E}(\mathbf{a}'\mathbf{M}\mathbf{b}) = \mathbf{a}'\mathrm{E}(\mathbf{M})\mathbf{b}$.

3. $\mathrm{Cov}(\mathbf{a}'\mathbf{X}, \mathbf{b}'\mathbf{X}) = \mathbf{a}'\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{b}$. (Lemma 5.3.3(a) is a special case.)

4. $\mathrm{Var}(\mathbf{a}'\mathbf{X}) = \mathbf{a}'\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{a}$. (Written out, this is Problem 4.44.)

5. Covariance matrices are positive (semidefinite): $\mathbf{a}'\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{a} \geq 0$, and $=0$ iff $\mathbf{a}'\mathbf{X} =$ a constant a.s. Thus $\Sigma_{\mathbf{X}}$ is positive *definite* (and therefore nonsingular) iff the only such $\mathbf{a}$ is $\mathbf{a} = \mathbf{0}$.

Def. The *moment generating function* of the vector $\mathbf{X}$ is $M(t_1, \ldots, t_n) = \mathrm{E}(e^{\mathbf{t}'\mathbf{X}})$, where $\mathbf{t}' = [t_1, \ldots, t_n]$.

Def.: The vector $\mathbf{X}$ has the *multivariate normal* distribution if every linear combination $\mathbf{a}'\mathbf{X} = a_1X_1 + \cdots a_nX_n$ has a (univariate) normal distribution. We write $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

Let $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and let $\mathbf{t}' = [t_1, \ldots, t_n]$. Then:

6. $M_{\mathbf{X}}(t_1, \ldots, t_n) = \exp(\boldsymbol{\mu}'\mathbf{t} + \mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}/2)$. (See Problem **A18**.) A multivariate normal distribution is completely determined by $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. It is degenerate if $\boldsymbol{\Sigma}$ is singular.

7. $\sigma_{ij} = 0$ implies that X_i and X_j are independent (the converse is true for any distribution).

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Here are some further results. **X** and **Y** are random vectors, and **A** denotes a constant matrix.

8. $\mathrm{E}(\mathbf{AX}) = \mathbf{A}\mathrm{E}(\mathbf{X})$. (This generalizes (1).)
9. $\boldsymbol{\Sigma}_{\mathbf{AX}} = \mathbf{A}\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{A}'$. (This generalizes (4).)
10. If **X** is normally distributed then so is $\mathbf{Y} = \mathbf{AX}$.

    This follows because $\mathbf{a}'\mathbf{Y} = \mathbf{a}'\mathbf{AX} = \mathbf{b}'\mathbf{X}$ for some vector **b**.
11. Let **A** be an *orthogonal matrix*; that is, **A** is s square matrix such that $\mathbf{A}'\mathbf{A} = \mathbf{I} = \mathbf{AA}'$. Let $\mathbf{Y} = \mathbf{AX}$. Then we have the following:
    (a) $\sum Y_i^2 = \sum X_i^2$.
    (b) If $\boldsymbol{\Sigma}_X = \sigma^2 I$ then $\boldsymbol{\Sigma}_Y = \sigma^2 I$.
12. If $Z_1, \ldots, Z_m$ are i.i.d $N(0,1)$, then $\sum_1^m Z_i^2 \sim \chi^2(m)$. (See Lemma 5.3.2.)