Stat 761

Jensen's Inequality

Theorem. *Let X be a random variable such that $a < X < b$ a.s. If g is convex on (a, b), then*

$$\mathrm{E}(g(X)) \geq g(\mathrm{E}(X)),$$

with equality iff either X is degenerate or g is linear on $(c, d) \subset (a, b)$, where $c = \operatorname{ess\,inf} X$ and $d = \operatorname{ess\,sup} X$.

Some technical notes:

- We allow $a = -\infty$ or $b = \infty$.
- Roughly, c is the minimum value and d the maximum value of X.

  More precisely, $\operatorname{ess\,sup} X$ (the *essential supremum of* X) is defined as follows. Let $S = \{x : \mathrm{P}(X > x) = 0\}$.

  If $S = \emptyset$, then $\operatorname{ess\,sup} X = \infty$.

  If $S \neq \emptyset$, then $\operatorname{ess\,sup} X = \inf S$.

  The *essential infimum* $\operatorname{ess\,inf} X$ is defined analogously.

- The proof of the theorem uses the positivity of E in the form

$$U > V \textit{ a.s.} \Rightarrow \mathrm{E}(U) > \mathrm{E}(V).$$

- The statement of the theorem requires that $g(X)$ be a random variable. Actually,

$$\begin{aligned} g \text{ convex on } (a,b) &\overset{*}{\Rightarrow} g \text{ continuous on } (a,b) \\ &\Rightarrow g \text{ measurable on } (a,b) \\ &\Rightarrow g(X) \text{ is a random variable.} \end{aligned}$$

  * requires that the interval be open. (Counterexample: the function g on [0,1] given by $g(x) = 0$ for $0 \leq x < 1$ and $g(1) = 1$ is convex but not continuous.) See standard texts on real analysis.

- The text (p. 190) frames the condition for "equality" in terms of the tangent line to g at x. But this presumes differentiability. However, this is a useful way to depict the theorem – see the graph on that page.