Stat 761 **Fall 2005**

Properties of Expectation

In the following (and in general) the abbreviation "a.s." stands for "almost surely" and means that a given statement holds "with probability one."

The first four properties are in Theorem 2.2.5.

1. **Linearity:** $\mathrm{E}(aY) = a\mathrm{E}(Y)$.
2. **Additivity:** $\mathrm{E}(Y + Z) = \mathrm{E}(Y) + \mathrm{E}(Z)$.
3. **Constant-preservation:** $\mathrm{E}(1) = 1$.
4. **Positivity:** If $Y \geq 0$ a.s., then $\mathrm{E}(Y) \geq 0$, with equality iff $Y = 0$ a.s.
5. **Contraction:** $|\mathrm{E}(Y)| \leq \mathrm{E}|Y|$, with equality iff either $Y \geq 0$ a.s. or $Y \leq 0$ a.s. (See Problem A10.)
6. **Least-squares property:** See equation (2.2.3).
7. **Probability property:** $\mathrm{P}(A) = \mathrm{E}(I_A)$. ($I_A$ is the indicator of the event A.)
8. Let X be a random variable with pdf/pmf $f(x)$, let $g(x)$ be a (measurable) function, and let $Y = g(X)$. Then
$$\mathrm{E}(Y) = \begin{cases} \sum_x g(x) f_X(x) \\ \int_{-\infty}^{\infty} g(x) f_X(x) dx \end{cases} \tag{1}$$
This result (see equation (2.2.5)) is sometimes called the Theorem of the Unconscious Statistician, as it is often used rather automatically and without thinking. Our text simply *defines* $\mathrm{E}(Y)$ by equation (1). The problem is that $\mathrm{E}(Y)$ also has its elementary definition (as $\sum y f_Y(y)$ or $\int y f_Y(y)$), and one needs to verify that the elementary definition and (1) give the same answer!