Stat 761 **Fall 2005**

Assignments for Chapter 4

Chapter	Section	Problems
4	1	5, A14(a-d)
	2	14a, 15, A14(e-g), A15
	3	13*, 19b*
	4	32a
	5	47, 58*, A16, A17
	6	A18, A19
	7	63, A20

Additional Problems:

A14: Let X, Y have joint pdf $f(x,y) = kxy,\ 0 < x < y < 1, = 0$ otherwise.

a. Show $k = 8$.

b. Show $f_Y(y) = 4y^3, y \in (0,1), = 0$ otherwise.

c. Find $\mathrm{P}(X + Y < 1/2)$.

d. Find the joint cdf $F(x,y)$. (Note: There are 5 cases to consider, depending on the location of (x,y). Equation (4.1.4) will provide a partial check of your answer.)

e. Find $f_{Y|x}$.

f. Find $\mathrm{P}(Y < 2/3 | X = 1/3)$.

g. Find $\mathrm{E}(Y|x)$ and $\mathrm{Var}(Y|x)$.

A15: Show that the random variables X and Y of Example 4.5.9 (pp.174-5) are stochastically dependent by finding sets A and B such that $\mathrm{P}(X \in A, Y \in B) \neq \mathrm{P}(X \in A)\mathrm{P}(Y \in B)$.

A16: Let N_t be the number of events occuring during the interval $(0,t)$ in a Poisson process.

a. Prove that $s < t \Rightarrow N_s$ and $N_t - N_s$ are independent random variables.

b. Compute $\mathrm{Cov}(N_s, N_t)$. (Hint: Add and subtract N_s in the case $s < t$.)

A17: (*Variance of hypergeometric.*) Let a_i, X_i and Y be defined as in **A12**. Find

a. $\mathrm{Var}(X_i)$.

b. $\mathrm{Cov}(X_i, X_j)$ for $i \neq j$.

c. $\mathrm{Var}(Y)$.

A18: Let $X_1, \ldots, X_n$ have a multivariate normal distribution *as defined in class* (i.e., such that $\mathbf{a}'\mathbf{X} = \sum_i a_i X_i$ is (univariate) normal for every choice of constants $a_1, \ldots, a_n$). Let $M(\mathbf{t}) = M(t_1, \ldots, t_n)$ be their joint mgf. Derive from this the formula for M in terms of the mean vector $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$. (Hint: Fix $\mathbf{t}$, and let $Z = \mathbf{t}'\mathbf{X} = \sum_i t_i X_i$. Show that $M_Z(1) = M(\mathbf{t})$, where M_Z is the mgf of Z. Then write down $M_Z(1)$ from what you know about the univariate normal mgf and the mean and variance of Z.)

A19: a. Let $X_1, \ldots, X_n$ be independent Poisson random variables with $\mathrm{E}(X_i) = \lambda_i$. Find the distribution of $Y = \sum_{i=1}^n X_i$.

b. Let X_1 and X_2 be independent random variables and let $Y = X_1 + X_2$. Suppose that $X_1 \sim \chi^2(r_1)$ and $Y \sim \chi^2(r)$, with $r_1 < r$. What is the distribution of X_2?

A20: (For full credit, do not assume r and s to be integers.)

a. Show that if μ'_r exists then μ'_s exists for $0 < s \le r$.
b. Show that μ_r exists iff μ'_r exists, $r \ge 1$.

***Notes, hints:**

Problem 4.13: Add and subtract $\mathrm{E}(Y|X)$ and use the composition and averaging properties of $\mathrm{E}(\cdot|\cdot)$. (For full credit, make no assumptions about the distributions of X and Y.) Note that part (b) refers to equation (2.2.3), not (2.2.4).

You may find it convenient to introduce $\hat{Y} = \mathrm{E}(Y|X)$.

Problem 4.19b: Do $X_1/(X_1 + X_2)$. Just state the result for $X_2/(X_1 + X_2)$.

Problem 4.58: You may again find it convenient to let $\hat{Y} = \mathrm{E}(Y|X)$.

Extra Credit Problems:

ECR3 Prove that the function $I_{(0,\theta)}(x)$ cannot be written as $h(x)c(\theta)$ for functions h and c defined on $\mathbb{R}$. (This shows that the uniform distribution on $(0, \theta)$ is not an exponential family.)

ECR4 Prove that the n-variate $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ pdf (for $\boldsymbol{\Sigma}$ nonsingular) is

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{1/2}} \exp(-(1/2)(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}))$$

by showing that it gives rise to the right mgf (problem **A18**).

ECR5 In Problem 4.47(b), find a and b (nonzero) such that $aZ + bY$ is non-normal. (This is an alternate way to prove part (b), according to our definition of joint normality.)

ECR6 Let X and Y be i.i.d. with common pdf $f(x) = xe^{-x^2/2}, x > 0$ (and 0 otherwise). Find $\mathrm{P}(X + Y < 1)$.

(This problem is given as example 5B on page 291 of E. Parzen, *Modern Probability Theory and Its Applications*. He simply gives the answer of .2433, but this must be wrong, since $\mathrm{P}(X + Y < 1) < \mathrm{P}(X < 1, Y < 1)$, which he correctly shows $= .1548$. The correct answer is $1 - e^{-1/2} - e^{-1/4}\sqrt{\pi}(\Phi(1/\sqrt{2}) - 1/2) \doteq .03422$, where Φ is the cdf of the $N(0, 1)$ distribution.)