MATH 731, FALL 2008
HOMEWORK SET 8 sample solutions

We will use the following fact.

Lemma 0. *Let N be a normal subgroup of G.*

(1) Every representation $\bar{\rho}$ of G/N lifts naturally to a representation ρ of G by $\rho(g) = \bar{\rho}(gN)$.

(2) If ρ is a representation of G and $N \subseteq \ker \rho$, then $\bar{\rho}$ defined by $\bar{\rho}(gN) = \rho(g)$ is a well-defined representation of G/N.

Moreover, in either (1) or (2), the original representation is irreducible iff the new representation is; two original representations are isomorphic iff the original ones are; the original and the new representation have the "same" character.

Proof. This is straightforward: the key fact is that the actions of G and G/N are the "same" if well-defined, so all properties are preserved. ■

A. *Prove that if V is an irreducible representation of G, then* $\dim V \le |G|$.

Proof. Let $v \in V$ be nonzero, and let W be the span of all $g.v$ with $g \in G$. This spanning set has at most $|G|$ elements, so $\dim W \le |G|$. We'll show W is an invariant subspace. If $w \in W$, it is a linear combination of elements $g.v$, whence $h.w$ is a linear combination of elements $(hg).v$, all of which are in W. Thus W is invariant and $W \neq \{0\}$, so we must have $W = V$ by irreducibility of V. This shows $\dim V \le |G|$.

B. *Prove that the following statements are equivalent.*
(1) The commutator subgroup $[G,G]$ equals G.
(2) The only 1-dimensional representation of G is the trivial representation.

Proof. (1) $\Rightarrow$ (2). Let $\chi : G \to GL_1(\mathbb{C})$ be an irreducible representation of G. If $x, y \in G$, then $\chi([x,y]) = \chi(x)\chi(y)\chi(x)^{-1}\chi(y)^{-1} = 1$ since $GL_1(\mathbb{C})$ is Abelian. As χ is a group homomorphism and its kernel contains all commutators, it must contain the subgroup $[G,G]$ they generate. Thus $\ker \chi = G$. This shows χ is trivial.

(2) $\Rightarrow$ (1). Suppose $[G,G]$ is a proper subgroup of G and set $A = G/[G,G]$. Then A is an Abelian group of order $n > 1$, so A has n distinct conjugacy classes. Corollary 7.14 and Proposition 7.2 imply that A has n inequivalent 1-dimensional representations. The lemma at the top tells us we can lift these to n inequivalent 1-dimensional representations of G. This contradicts our original assumption, so we must have $[G,G] = G$.

C. *Suppose g is in the center $Z(G)$ and suppose V is an irreducible representation of G. Show that there is a scalar $\lambda \in \mathbb{C}$ such that $g.v = \lambda v$ for every $v \in V$.*

Proof. Let ρ be the representation. The linear map $\rho(g)$ has at least one eigenvalue λ. Set $W = \{\, v \in V \mid g.v = \lambda v \,\}$. Then W is not zero since λ is an eigenvalue.

We claim W is an invariant subspace. Once we show this, it will follows that $W = V$ and we will be done. To see W is invariant, let $w \in W$, $h \in G$. Then $g.(h.w) = (gh).w = (hg).w = h.(g.w) = h.(\lambda w) = \lambda h.w$. Thus $h.w \in W$, so W is invariant.

The above copies the proof of Schur's Lemma. One can also apply Schur's Lemma itself as follows. Define $T : V \to V$ by $T = \rho(g)$, that is, $T(v) = g.v$. Then for any $h \in G$, we have $T(h.v) = g.(h.v) = gh.v = h.(g.v)$, so T is a homomorphism. By Schur's Lemma, there is a $\lambda \in \mathbb{C}$ with $T(v) = \lambda v$ for all v.

D. *Show that if every irreducible representation of G is 1-dimensional, then G is Abelian.*

Proof. Let $|G| = n$ and let k be the number of conjugacy classes in G. By Proposition 7.12 and Corollary 7.14, G has k irreducible representations and the sum of the squares of their dimensions is n. But each dimension is 1, so this sum is k. Thus $k = n$, i.e., every element of G is its own conjugacy class. This is equivalent to saying G is Abelian.

Here is another proof. If $\rho : G \to GL_1(\mathbb{C})$ is a 1-dimensional representation, then $\rho(gh) = \rho(g)\rho(h) = \rho(hg)$ for all $g, h \in G$. This implies $\rho(gh) = \rho(hg)$ for any representation that is a direct sum of 1-dimensional representations. If G has only 1-dimensional irreducible representations, it follows that $\rho(gh) = \rho(hg)$ for any representation.

If we apply this observation to the regular representation, we see $v_{gh} = (gh).v_e = (hg).v_e = v_{hg}$ for all g, h. It follows that $gh = hg$ for all g, h.

E. *Let G be the subgroup of $GL_2(\mathbb{C})$ containing all matrices of the form $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$. Here all possible combinations of ± are allowed, so $|G| = 8$.*
It is a fact that G has 5 irreducible representations, up to isomorphism. These representations are listed below, with partial information. Verify that the representations are indeed irreducible and fill in any missing information (such as the dimension and the character), with justification.
(1) The trivial representation ($\dim V = 1$, $\chi(g) = 1$ for all g).
(2) The determinant representation ($\dim V = 1$, $\chi(g) = \det g$ for all g).
(3) The natural representation of G on the vector space $V = \mathbb{C}^2$ of column vectors. ("Natural" means $\rho : G \to GL_2(\mathbb{C})$ is the inclusion map.)
(4) Irreducible representation #4.
(5) Irreducible representation #5.

Answers & Proofs. There is nothing to prove in (1) and (2): the representations are completely described and they are irreducible since their dimensions are 1. These representations are not isomorphic, because they have different characters.

The representation in (3) is 2-dimensional and the character is given by $\chi(I) = 2$, $\chi(-I) = -2$, $\chi(g) = 0$ for all other g.

To see V is irreducible, suppose W is a proper nonzero invariant subspace, i.e., W is 1-dimensional. If W contains a vector $\begin{pmatrix} a \\ b \end{pmatrix}$ with $b \neq \pm a$, then W also contains $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$. Since $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} b \\ a \end{pmatrix}$ are linearly independent, this is impossible.

If W contains a nonzero vector of the form $\begin{pmatrix} a \\ a \end{pmatrix}$, then $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ a \end{pmatrix} = \begin{pmatrix} a \\ -a \end{pmatrix}$ is also in W and these two vectors are linearly independent, an impossibility.

Finally, if W contains a nonzero vector of the form $\begin{pmatrix} a \\ -a \end{pmatrix}$, then $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ -a \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix}$ is also in W and these two vectors are linearly independent, an impossibility.

Thus such a W cannot exist.

A simpler way to show W is irreducible is to compute $\langle \chi, \chi \rangle = \dfrac{1}{8}\left(2^2 + (-2)^2 + 0 + \cdots + 0\right) = 1$. This shows W can have only one irreducible summand, so it is irreducible.

We now know $8 = |G| = 1^2 + 1^2 + 2^2 + n_4^2 + n_5^2 + \cdots$. The only way to solve this equation is to have $n_4 = n_5 = 1$ and no further representations. Thus the final two representations are 1-dimensional, and to describe them, we simply have to give the homomorphisms $\chi : G \to GL_1(\mathbb{C})$ that define them.

To find the 1-dimensional representations, we need to understand G a little better. Observe that the center of G is $\{I, -I\}$. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then A has order 4 and P has order 2, and a complete list of elements of G is $I, A, A^2 = -I, A^3 = -A, P, -P, AP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, PA = -AP$.

The 1-dimensional representations are precisely the 1-dimensional representations of $G/[G,G]$. Looking at the list of members of G above, we can see that $[G,G]$ will be generated by $[A,P] = APA^{-1}P^{-1} = AP(-A)P = APPA = A^2 = -I$. Thus $[G,G] = Z(G) = \{\pm I\}$. The quotient $G/[G,G]$ is generated by the cosets of A and P, so it is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. The four 1-dimensional representations are realized by (i) $\chi(A) = \chi(P) = 1$ (the trivial representation); (ii) $\chi(A) = 1$, $\chi(P) = -1$ (the determinant representation); (iii) $\chi(A) = -1$, $\chi(P) = 1$; (iv) $\chi(A) = \chi(P) = -1$. Representations (iii) and (iv) are the final two irreducible representations we were seeking.

Representation (iii) is given explicitly by $\chi(g) = 1$ for $g = \pm I, \pm P$ and $\chi(g) = -1$ for other g.

Representation (iv) is given explicitly by $\chi(g) = -1$ for $g = \pm A, \pm P$ and $\chi(g) = 1$ for other g.