# MATH 731, FALL 2008
# HOMEWORK SET 7

Due Friday, October 31 at noon

A. Suppose $A, B, C$ are subgroups of $G$ with $A \triangleleft B$, $C \triangleleft G$ and suppose $BC = G$. Prove that $AC$ is a normal subgroup of $G$.

B. Suppose $G$ is an Abelian group that acts transitively on the set $X$.

(1) Show that $G$ acts faithfully if and only if $g.x = x$ only happens when $g = e$ (regardless of what $x$ is). [Such an action is called **free**.]

(2) Show that $G$ acts doubly transitively and faithfully on $X$ if and only if $|G| = |X| = 2$ and the non-identity element of $G$ swaps the two elements of $X$.
[Hint: Let $x \in X$ and let $g_1, g_2$ be elements of $G$ different from the identity – possibly $g_1 = g_2$ – and consider the "source pair" $x, g_2.x$ and the "target pair" $g_1.x, x$ in the definition of doubly transitive.]

C. (1) Prove that if $n \geq 3$, every element of $A_n$ is a product of 3-cycles. [This is true for all $n$ if we understand the product of zero 3-cycles to be the identity.]

(2) Prove that if $n \geq 5$, every element of $A_n$ is a product of permutations of the form $(a\,b)(c\,d)$ where $a, b, c, d$ are distinct.

D. Suppose $F$ is a finite field with $q$ elements (so $F = \mathbb{F}_q$). Explain the following equalities.

(1) $|\mathbb{P}^{n-1}(F)| = (q^n - 1)/(q - 1)$.

(2) $|GL_n(F)| = \prod_{i=0}^{n-1}(q^n - q^i) = q^{n(n-1)/2} \cdot (q^n - 1)(q^{n-1} - 1) \cdots (q - 1)$.