MATH 731, FALL 2008
HOMEWORK SET 3 sample solutions

A. *(i) Let V be the vector space of continuous complex-valued functions with domain $[0,1]$. Show V **does not** become an inner product space when we define $\langle f,g\rangle = \int_0^1 f(x)g(x)\,dx$ for $f,g \in V$.*

Proof. Let $f(x) = ix$. Then $\langle f,f\rangle = \int_0^1 f(x)^2\,dx = \int_0^1 -x^2\,dx = -\dfrac{1}{3} < 0$. Thus $\langle\,,\rangle$ is not positive definite and hence not an inner product.

(ii) Explain how to alter (in a simple way) the definition of $\langle f,g\rangle$ in (i) to obtain an inner product.

Just as in the dot product case, we redefine $\langle f,g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$.

(iii) *Verify that your altered definition does in fact yield an inner product.*

Proof. Linearity: $\langle f_1{+}f_2,g\rangle = \int_0^1 (f_1{+}f_2)(x)\overline{g(x)}\,dx = \int_0^1 f_1(x)\overline{g(x)}\,dx + \int_0^1 f_2(x)\overline{g(x)}\,dx = \langle f_1,g\rangle + \langle f_2,g\rangle$ and $\langle cf,g\rangle = \int_0^1 cf(x)\overline{g(x)}\,dx = c\int_0^1 f(x)\overline{g(x)} = c\langle f,g\rangle$.

Hermitian symmetry: $\langle g,f\rangle = \int_0^1 g(x)\overline{f(x)}\,dx = \int_0^1 \overline{f(x)\overline{g(x)}}\,dx = \overline{\int_0^1 f(x)\overline{g(x)}\,dx} = \overline{\langle f,g\rangle}$.

Positive definiteness: Suppose $f \neq 0$. Since f is continuous, there exists an interval I of length $\ell(I) > 0$ and a $\delta > 0$ such that $|f(x)| \geq \delta$ for all $x \in I$. Thus $\langle f,f\rangle = \int_0^1 |f(x)|^2\,dx \geq \int_I \delta^2 = \ell(I)\delta^2 > 0$.

B. *Let $\langle\,,\rangle$ be an inner product on the vector space V and define $\|\cdot\| : V \to \mathbb{R}$ by $\|v\| = \sqrt{\langle v,v\rangle}$. Prove that $\|\cdot\| : V \to \mathbb{R}$ is a norm, that is, prove the following three properties hold.*
(1) $\|v\| > 0$ for $v \in V, v \neq 0$; *(2) $\|cv\| = |c|\|v\|$ for $c \in F, v \in V$;*
(3) $\|v+w\| \leq \|v\| + \|w\|$ for all $v,w \in V$.

Proof. First we prove the Cauchy-Schwartz inequality. What follows is one of many possible proofs.
If $w = 0$, both sides are 0, so we may assume $w \neq 0$. Let $v,w \in V$, set $c = -\langle v,w\rangle/\langle w,w\rangle$ and let $x = v + cw$. Then

$$\begin{aligned}
0 \leq \langle x,x\rangle =& \langle v,v\rangle + c\langle w,v\rangle + \bar{c}\langle v,w\rangle + c\bar{c}\langle w,w\rangle = \\
=& \|v\|^2 + |c|^2\|w\|^2 - \langle v,w\rangle\overline{\langle v,w\rangle}/\|w\|^2 - \overline{\langle v,w\rangle}\langle v,w\rangle/\|w\|^2 = \\
=& \|v\|^2 + \|w\|^2|\langle v,w\rangle|^2/\|w\|^4 - 2|\langle v,w\rangle|^2/\|w\|^2 = \\
=& \|v\|^2 + |\langle v,w\rangle|^2/\|w\|^2 - 2|\langle v,w\rangle|^2/\|w\|^2.
\end{aligned}$$

Clearing denominators leads to $0 \leq \|v\|^2\|w\|^2 - |\langle v,w\rangle|^2$, i.e., $|\langle v,w\rangle|^2 \leq \|v\|^2\|w\|^2$. This is what we needed to prove.

We now proceed to prove the norm properties. Condition (1) is immediate: $\|v\| = \sqrt{\langle v,v\rangle} > 0$ if $v \neq 0$ by positive-definiteness of $\langle\,,\rangle$.

Condition (2) is also easy: $\|cv\| = \sqrt{\langle cv,cv\rangle} = \sqrt{c\bar{c}\langle v,v\rangle} = \sqrt{|c|^2\|v\|^2} = |c|\|v\|$.

To prove (3), note that $\|v+w\|^2 = \langle v+w,v+w\rangle = \langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle = \|v\|^2 + 2\cdot\text{real part}(\langle v,w\rangle) + \|w\|^2 \leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 = (\|v\| + \|w\|)^2$, where the $\leq$ follows from the Cauchy-Schwartz Inequality. Thus $\|v+w\| \leq \|v\| + \|w\|$.