

**MATH 731, FALL 2008**  
**HOMEWORK SET 1**

Originally due Wednesday, September 10 – due date changed to Friday, September 12

- A. Let  $A, B$  be arbitrary sets and let  $f : A \rightarrow B$  be a function. Prove that the following statements are equivalent.
- (1)  $f$  is onto.
  - (2)  $f$  is right invertible, that is, there exists  $g : B \rightarrow A$  with  $f \circ g = \text{id}_B$ .
  - (3)  $f$  is right cancellable, that is, if  $g, h : B \rightarrow C$  are any functions with  $g \circ f = h \circ f$ , then  $g = h$ .
- B. (i) Show that a group  $G$  is Abelian if and only if the map  $\phi : G \rightarrow G$  defined by  $\phi(g) = g^{-1}$  is a group homomorphism.  
(ii) Determine for which fields  $F$  the map  $\phi : F \rightarrow F$  defined by  $\phi(0) = 0$  and  $\phi(x) = x^{-1}$  for  $x \neq 0$  is a field homomorphism.
- C. Let  $V$  be a vector space and let  $\pi : V \rightarrow V$  be a linear operator. We say  $\pi$  is an **internal projection** if  $\pi^2 = \pi$ . (The word “internal” is often omitted.)  
(i) Let  $\pi : V \rightarrow V$  be an internal projection. Prove that  $V = \text{im } \pi \oplus \ker \pi$ .  
(ii) Suppose  $W, W'$  are subspaces of  $V$  with  $V = W \oplus W'$ . Prove that there is an internal projection  $\pi : V \rightarrow V$  with  $\text{im } \pi = W$ ,  $\pi|_W = \text{id}_W$ , and  $\ker \pi = W'$ .
- D. Let  $V$  be a vector space over the field  $F$  and let  $U, W$  be subspaces. Prove that  $\dim(U + W) + \dim(U \cap W) = \dim U + \dim W$ .
- E. Find all  $2 \times 2$  matrices  $A$  over  $\mathbb{R}$  satisfying  $A^2 = -I$ .  
Hints: There are many such matrices. First show that  $x^2 + 1$  must be both the minimal and characteristic polynomial of  $A$ . The “characteristic” fact gives you the trace and determinant of  $A$ .

**Notes:** B(ii) is probably hard - it depends on some knowledge of fields.

For C, recall that  $V = W \oplus W'$  means  $V = W + W'$  and  $W \cap W' = \{0\}$ .

D can be done by using the Dimension Formula and the First Isomorphism Theorem.