

MATH 632 Homework List 2 11 February 2000

Sections 2 & 3: Symmetry and Group Actions

The following problems give you practice with the material. I will ask you to hand in *some* of them for a grade later. I may also assign additional problems.

*Problems* from Chapter 5 of Artin, pp. 188–196

- §1 #1,2b
- §2 #2,3,6,15b
- §3 #2,4,5
- §5 #3,5
- §6 #1,3a,4a
- §7 #1,3,4
- §8 #2

*Problems* from Chapter 6 of Artin, pp. 229–230

- §1 #2,10af,12,15

*Problems* from the class notes (Sections 2 and 3 on the web page)

**Problem 2.A.** This is a problem partly because it takes some effort to create the appropriate drawings in  $\text{\TeX}$ . We will discuss this problem in class: prepare for it!

Draw as many examples of triangles as you can that exhibit *different* amounts/levels of symmetry. (There are 3.)

Draw as many examples of quadrilaterals (4-sided polygons) as you can that exhibit *different* amounts/levels/kinds of symmetry. (Are there 6?)

**Problem 2.B.** The maps  $\psi$  and  $\psi'$  in Theorem 2.1 are actually the same. Show this by proving that for any linear map  $\phi$  and any  $\mathbf{p} \in \mathbb{R}^n$ , we have  $\phi \circ \tau_{\mathbf{p}} = \tau_{\phi(\mathbf{p})} \circ \phi$ .

**Problem 2.C.** If  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , then  $\phi$  is linear iff for any  $c, d \in \mathbb{R}$  and  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ , we have  $\phi(c\mathbf{x} + d\mathbf{y}) = c\phi(\mathbf{x}) + d\phi(\mathbf{y})$ . This equality does not hold for all symmetries, as the example of translation shows. But symmetries do possess a weakened form of linearity.

Prove that if  $\phi$  is a rigid motion, then  $\phi(c\mathbf{x} + d\mathbf{y}) = c\phi(\mathbf{x}) + d\phi(\mathbf{y})$  holds whenever  $c + d = 1$ . This can be proved either geometrically (the points  $c\mathbf{x} + d\mathbf{y}$  with  $c + d = 1$  make up the line through  $\mathbf{x}$  and  $\mathbf{y}$ ) or by using Theorem 2.1 (or Corollary 2.3).

**Problem 2.D.** If  $\ell$  is the line through the origin making an angle  $\phi$  with the positive  $x$ -axis, show that reflection about  $\ell$  is the linear map corresponding to the matrix

$$\begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}.$$

**Problem 2.E.** Let  $n \geq 3$ , let  $\rho$  be rotation about the origin by  $2\pi/n$ , and let  $r$  be any reflection in  $D_n$ .

- (1) Show that  $D_n$  consists of the elements  $\rho^j, \rho^k \circ r, 0 \leq j, k < n$ , and that this list contains no duplicates.
- (2) Show that  $D_n$  consists of the elements  $\rho^j, r \circ \rho^k, 0 \leq j, k < n$ , and that this list contains no duplicates.
- (3) [You may want to do this first, or not do it at all.] Find formulas for the products  $\rho_\theta \circ r_\ell$  and  $r_\ell \circ \rho_\theta$  where  $\ell$  is a line through the origin.

**Problem 2.F.** Let  $\mathbf{p} \in \mathbb{R}^n$  and let  $G$  be a subgroup of  $\mathcal{M}_n$  such that  $\phi(\mathbf{p}) = \mathbf{p}$  for all  $\phi \in G$ . If  $\tau = \tau_{\mathbf{p}}$  is translation by  $\mathbf{p}$ , prove that  $H = \tau^{-1}G\tau$  is a subgroup of  $\mathcal{M}_n$  such that  $\phi'(\mathbf{0}) = \mathbf{0}$  for all  $\phi' \in H$ .

(over)

**Problem 3.A.** Let  $G$  be a group and  $X$  a set. Let  $S(X)$  denote the permutation group of  $X$ , that is, the set of all bijections from  $X$  to  $X$  with composition as operation.

- (1) Suppose  $G$  acts on  $X$ . For each  $g \in G$ , define  $\phi_g : X \rightarrow X$  by  $\phi_g(x) = g.x$ . Define  $\Phi : G \rightarrow S(X)$  by  $\Phi(g) = \phi_g$ . Show that  $\Phi$  is a group homomorphism. (Note: one thing you need to show is that the image of  $\Phi$  actually is contained in  $S(X)$ , since it is not obvious that each  $\phi_g$  is a bijection.)
- (2) Let  $\Phi : G \rightarrow S(X)$  be a homomorphism of groups. For  $g \in G, x \in X$ , define  $g.x = \Phi(g)(x)$ . Show that this defines a group action of  $G$  on  $X$ .
- (3) In the first two parts of this problem we have defined maps between the set of group actions of  $G$  on  $X$  and the set of group homomorphisms from  $G$  to  $S(X)$ . Show that these maps define a bijective correspondence. More precisely, these maps are inverses of each other.

**Problem 3.B.** Let  $G$  be a group with  $n$  elements. We can define an action of  $G$  on itself (different from the conjugation action defined in Example 3.1(5)) by simple multiplication:  $g.h = gh$ . Since  $|G| = n$ , Problem 3.A implies that this action determines a homomorphism  $\Phi : G \rightarrow S_n$ . Show that this homomorphism is one-to-one and conclude that  $G$  is isomorphic to a subgroup of  $S_n$ .

Thus every finite group is isomorphic to a subgroup of some  $S_n$ .

**Problem 3.C.** Let  $p$  be a prime number. Show that every group of order  $p^2$  is abelian.