MATH 632 Homework List 1 *24 January 2000*

Section 1: Inner Product Spaces

The following problems give you practice with the material on inner product spaces. I will ask you to hand in *some* of them for a grade later. I may also assign additional problems.

Problems from Chapter 7 of `Artin`, pp. 262–269
Note: What `Artin` calls either a "positive definite bilinear form" (when the field F is $\mathbb{R}$) or a "positive definite hermitian form" (when $F = \mathbb{C}$), we call an *inner product*.

§1 #3
§2 #4,5,9
§4 #6,7,8,10,16,18,19
§5 #2,3,4,5,14(hard)
§7 #4,5,9
§9 #3(i)-(iv)

Problems from the class notes (Section 1 on the web page)
Note: The fields F we deal with satisfy: (1) F is a subfield of $\mathbb{C}$; (2) F is closed under complex conjugation; (3) every positive element of F has a square root in F.

Problem 1.A. (1) Let P_n be the vector space over $\mathbb{R}$ of real polynomials of degree at most n, and let $P = \mathbb{R}[x]$. Show P_n and P become inner product spaces when we define $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$ for polynomials f, g.

(2) Show that we can replace $[0, 1]$ with any finite interval $[a, b]$ where $a < b$.

(3) Replace the field $\mathbb{R}$ with the field $\mathbb{C}$, so that P_n, P consist of complex polynomials. Show that the formula $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$ no longer defines an inner product. Explain how to alter the definition of $\langle f, g\rangle$ to obtain an inner product in this case.

Problem 1.B. The inner product $\langle\,,\rangle$ can actually be determined from the length $\|\cdot\|$. If $F \subseteq \mathbb{R}$, show that $\langle v, w\rangle = \frac{1}{2}(\|v + w\|^2 - \|v\|^2 - \|w\|^2)$. (This is the law of cosines.) This formula is called *polarization*. If $F \not\subseteq \mathbb{R}$, the formula for polarization is more complicated. If $F = \mathbb{C}$, show that $\langle v, w\rangle = \frac{1}{2}\left(\|v + w\|^2 + i\|v + iw\|^2 - (1 + i)(\|v\|^2 + \|w\|^2)\right)$.

Problem 1.C. Let us call a function $\|\cdot\| : F \to F$ a *norm* or *length function* on V if
(a) $\|v\| > 0$ for $v \in V, v \neq 0$; (b) $\|cv\| = |c|\|v\|$ for $c \in F, v \in V$; and
(c) [Triangle Inequality] $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

Assume that $F \subseteq \mathbb{R}$.
Define $\langle\,,\rangle$ on V by the polarization formula $\langle v, w\rangle = \frac{1}{2}(\|v + w\|^2 - \|v\|^2 - \|w\|^2)$.
Show that $\langle\,,\rangle$ is an inner product iff $\|\cdot\|$ satisfies the *parallelogram rule*:
$\|v + w\|^2 + \|v - w\|^2 = \|v\|^2 + \|w\|^2$ for all $v, w \in V$.

Problem 1.D. Let $\phi : V \to W$ be a linear map between inner product spaces. Show that ϕ is an isometry iff ϕ is onto and $\|\phi(v)\|_W = \|v\|_V$ for all $v \in V$. To simplify your work, you may assume $F \subseteq \mathbb{R}$ if you wish.

Problem 1.E. (1) Show that the set of unitary matrices forms a subgroup of $GL_n(F)$ for the fields F we are discussing.

(2) Show that for any field F, the set of orthogonal matrices forms a subgroup of $GL_n(F)$.

Problem 1.F. Find all real 2×2 normal matrices.

Problem 1.G. Let ϕ be a normal linear operator on the inner product space $\langle\,,\rangle$ and let v, w be eigenvectors of ϕ with different eigenvalues. Show that $\langle v, w\rangle = 0$.