MATH 631: Solution to Homework 1, Problem 1

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Let n and m be positive integers. Let A be the $n \times n$ matrix with 1's on and above the diagonal, and zeroes below the diagonal. Thus $a_{ij} = 1$ if $i \le j$ and $a_{ij} = 0$ if $i > j$. We can express this differently as follows.

If $a, b \in \mathbb{Z}$, we define the binomial coefficient $\binom{a}{b}$ to be 0 if $b < 0$ or $b > a$. If $0 \le b \le a$, then $\binom{a}{b} = \dfrac{a!}{b!(a-b)!}$, as usual (where $0! = 1$). With this notation, we can write $a_{ij} = \binom{j-i}{j-i}$.

There is an extremely important identity we will need below.

$$\binom{a+1}{b} = \binom{a}{b} + \binom{a}{b-1} \qquad \text{for all } a, b \in \mathbb{Z}.$$

Answer: Let m be a positive integer and set $B = A^m$. Then $b_{ij} = \dbinom{m-1+j-i}{j-i}$. Note that this formula tells us that B is an upper triangular matrix with 1's on the diagonal, so it is at least part right! Also note that the formula does not depend on n, which is a bit of a surprise.

Proof: We will prove the formula by induction on m. The formula is correct for $m = 1$, as we noted above.

Set $C = A^{m+1} = BA$. Then

$$\begin{aligned}
c_{ij} &= \sum_{k=1}^{m} b_{ik}a_{kj} = \sum_{k=1}^{j} b_{ik} = \sum_{k=i}^{j} \binom{m-1+k-i}{k-i} = \\
&= [\text{set } \ell = k-i, r = j-i] \quad \sum_{\ell=0}^{r} \binom{m-1+\ell}{\ell} = \\
&= [\text{use the identity}] \quad \sum_{\ell=0}^{r} \left(\binom{m+\ell}{\ell} - \binom{m-1+\ell}{\ell-1} \right) = \\
&= [\text{the sum telescopes}] \quad \binom{m+r}{r} - \binom{m-1}{-1} = \binom{(m+1)-1+j-i}{j-i)},
\end{aligned}$$

as required.