6. Odds and Ends from Group Theory

In this section we mention some topics presented in `Artin`, which we have not mentioned in the notes so far. We will not mention all of the topics from Chapter 2 of `Artin`, however.

One could say the theme of this section is "creating new groups from old". If G, H are groups, then the Cartesian product $G \times H = \{ (g,h) \mid g \in G, h \in H \}$ becomes a group with componentwise operations, that is, $(g,h)(g',h') = (gg', hh')$. This is called the *product group* or *direct product* of G and H.

More generally, if $G_1, \ldots, G_n$ are groups, the Cartesian product $G_1 \times \cdots \times G_n$ becomes a group with componentwise operations. Even more generally, one can define products of infinite collections of groups.

Actually, there is a clever way to define finite and infinite products. Let A be a set, and suppose for each $a \in A$, we are given a group G_a. Then the Cartesian product $\prod_{a \in A} G_a$ can be defined as the set H of all functions $f : A \to \cup_{a \in A} G_a$ with the property that $f(a) \in G_a$ for all $a \in A$. This product H becomes a group when we define ff' for $f, f' \in H$ by $(ff')(a) = f(a)f'(a)$ for $a \in A$.

Here a simple example. Let G be the Klein four-group, consisting of the 4 matrices $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$. Then $C_2 \times C_2 \cong G$ via the isomorphism $(a,b) \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ (here $C_2 = \{1, -1\}$).

The above construction is an external construction: it takes separate groups and creates a new group. We will now give some internal constructions which take subgroups or subsets of a given group and create a new subgroup.

The first such construction is the intersection.

Lemma 6.1. *Let G be a group, let A be a set, and let G_a be a collection of subgroups indexed by A. Then $\cap_{a \in A} G_a$ is a subgroup of G. If each $G_a \lhd G$, then $\cap_{a \in A} G_a \lhd G$.*

Remark. For example, if H, K are subgroups of G, then $H \cap K$ is a subgroup of G.

Proof. Exercise. ∎

Suppose X is a subset of a group G. When X consists of a single element, we have defined the cyclic subgroup of G generated by that element. Is there a general notion for arbitrary X? There are two and they coincide. Let us define the *subgroup generated by* X to be the subset of G consisting of all finite products of integer powers of elements of X, or equivalently, all elements of the form $x_1 x_2 \cdots x_n$ where n is arbitrary and for each i, either $x_i \in X$ or $x_i^{-1} \in X$. If $X = \emptyset$, we define the subgroup generated by X to be $\{e\}$. We denote the subgroup generated by X by $\langle X \rangle$.

Lemma 6.2. *Let X be a subset of the group G and let $H = \langle X \rangle$ be the subgroup of G generated by X, as just defined. Then H is the smallest subgroup of G containing X. That is, (a) $X \subseteq G$, (b) H is a subgroup of G, and (c) if H' is a subgroup of G with $X \subseteq H'$, then $H \subseteq H'$.*

Proof. Exercise. ∎

If $X = \{x, y\}$ and G is Abelian, then $\langle x, y \rangle = \{ x^i y^j \mid i, j \in \mathbb{Z} \}$. Thus for example in $\mathbb{Z}$, the subgroup generated by 4 and 6 is $4\mathbb{Z} + 6\mathbb{Z} = 2\mathbb{Z}$. If G is not Abelian, however, it is much more difficult to directly describe $\langle x, y \rangle$. It will contain elements like $y^{-1}x^3yx^{-1}y^4$, and it may not be possible to reduce this to a simpler form.

Suppose H, K are subgroups of G and one of them is normal. Then we can show (*verify!*) that $HK = KH$, and that this set is a subgroup. More generally, if $H_1, \ldots, H_n$

are subgroups of G and all (or all but one of them) are normal, then the product set is the same however we order them, and this set $H_1 H_2 \cdots H_n$ is a subgroup of G. If each $H_i \triangleleft G$, then $H_1 H_2 \cdots H_n \triangleleft G$.

Lemma 6.3. *Let G be a group and H, K be subgroups. Then the map $\phi : H \times K \to G$ is an isomorphism of groups if and only if the following 3 conditions hold.*

(1) $H, K \triangleleft G$;
(2) $H \cap K = \{e\}$;
(3) $HK = G$.

Proof. Exercise. ∎

Problem 6.A. Generalize Lemma 6.3 to more than 2 subgroups. Condition (2) needs to be appropriately modified.

Let us mention one more way of obtaining groups. If $\phi : G \to G'$ is a function of sets and $X \subseteq G$, $Y \subseteq G'$, we will write $\overrightarrow{\phi}(X) = \{\, \phi(x) \mid x \in X \,\} \subseteq G'$ and $\overleftarrow{\phi}(Y) = \{\, g \in G \mid \phi(g) \in Y \,\} \subseteq G$. These sets are called respectively the *direct image* of X under ϕ and the *inverse image* of Y under ϕ. Our notation is not standard: usually (for example, in `Artin`) one writes $\phi(X)$ instead of $\overrightarrow{\phi}(X)$ and $\phi^{-1}(Y)$ instead of $\overleftarrow{\phi}(Y)$. Note that we have defined two new functions $\overrightarrow{\phi} : \mathcal{P}(G) \to \mathcal{P}(G')$ and $\overleftarrow{\phi} : \mathcal{P}(G') \to \mathcal{P}(G)$

Thus for example if ϕ is a homomorphism of groups, $\operatorname{im}\phi = \overrightarrow{\phi}(G)$, while $\ker\phi = \overleftarrow{\phi}(\{e\})$. (We usually write this last set as $\overleftarrow{\phi}(e)$ or $\phi^{-1}(e)$: this last notation does not imply that a function ϕ^{-1} exists.)

Lemma 6.4. *Let $\phi : G \to G'$ be a homomorphism of groups. If H is a subgroup of G, then $\overrightarrow{\phi}(H)$ is subgroup of G'. If H' is a subgroup of G', then $\overleftarrow{\phi}(H')$ is a subgroup of G.*

Proof. Exercise. ∎

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