MATH 241 Sample Exam 1 October 15, 2001

This is a sample of what the exam on Monday, October 22 might look like. This is *not meant* to be exactly like the exam.
Use this sample to study by first attempting to do the problems as if you were taking the exam in class. Then, write them up well as if you were going to turn them in for a homework assignment. Both procedures will help you prepare for the exam. This sample is longer than the actual exam will be.

The first three problems require you to redo something covered in notes and in class – you must do this from scratch and not simply quote the corresponding results from the notes. The other problems are new; you may use results from the notes covered in class to help solve these problems.

1. Let A, B, C be sets. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

2. Let X be a set and let $f : X \to Y$, $g : Y \to Z$ be functions and assume that g is injective. Prove (**from scratch**) that $g \circ f$ is injective if and only if f is injective.

3. Let A, B, C be sets. Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

4. Suppose that x is an (arbitrary) integer. Prove that if $x + x + x$ is odd, then x is odd. What type of proof did you use?
   You may use basic facts about arithmetic and the integers in this problem.

5. Prove or give a counterexample to the following statement about arbitrary sets A, B:
   $$\mathcal{P}(A - B) = \mathcal{P}(A) - \mathcal{P}(B).$$

6. In this problem, the variables x, y denote planets. Let $P(x)$ be the open sentence "x is between Venus and Mars". Is the following statement true or false? Be sure to *explain/justify* your answer.
   $$(\exists x)\,(P(x) \text{ and } (\forall y)(P(y) \Rightarrow x = y))\,.$$

7. Let A, B, C be sets. Prove or disprove the following claim:
   $$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

8. Let A, B, C be sets. Suppose that $A \cup C \subseteq B \cup C$ and $A \cap C \subseteq B \cap C$. Prove that $A \subseteq B$.
   [Do not make any assumptions about the sets A, B, C except what you are given.]

9. Let $A = \{ 11, 22, 33, 44, 55, 66, 77, 88, 99 \}$ and let $B = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \}$. Define $f : A \to B$ by $f(a) =$ the first digit of a. Answer the following questions. Be sure to give a brief explanation/justification for your answers.

   (a) What are the domain and codomain of f?

   (b) Is f injective?

   (c) Is f surjective?

   (d) Is f bijective?

10. (a) Let P, Q, R be statements. Explain why the following statement is true. (You may use a truth table if you wish, or you may simply give an explanation in English.)

$$[P \Rightarrow (Q \text{ and } R)] \iff [(P \Rightarrow Q) \text{ and } (P \Rightarrow R)]$$

   (b) Let P be the sentence "$x \in A$", let Q be the sentence "$x \in B$", and let R be the sentence "$x \in C$". Express the statement you proved in (a) as a statement about the sets A, B, C. (It is a theorem, involving $\subseteq$.)

11. Let $f : A \to B$ be a function, and let $S, T \subseteq A$. Prove that

$$\{ f(x) \mid x \in S \cup T \} = \{ f(x) \mid x \in S \} \cup \{ f(x) \mid x \in T \}.$$