Problem Set 3 Solutions

1(a) let $\begin{pmatrix} u_n(r) \\ v_n(r) \end{pmatrix} = \begin{pmatrix} u_k \\ v_k \end{pmatrix} e^{i\underline{k}\cdot\underline{r}}$ since we have

translational invariance, then

$$\begin{pmatrix} \frac{\hbar^2 k^2}{2m} - \mu & \Delta \\ \Delta^+ & -\frac{\hbar^2 k^2}{2m} + \mu \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} = E_k \begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

- let $\xi_k = -\frac{\hbar^2 k^2}{2m} + \mu$, so that we have

$$\begin{pmatrix} -\xi_k & \Delta \\ \Delta^+ & +\xi_k \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} = E_k \begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

- find the E_k by diagonalizing:

$$(\lambda - \xi_k)(\lambda + \xi_k) - |\Delta|^2 = 0 \quad \Rightarrow \quad \lambda = E_k = \pm\sqrt{\xi_k^2 + |\Delta|^2}$$

- next find the eigenvectors:

$$(E_k + \xi_k) u_k - \Delta v_k = 0$$

$\Rightarrow \quad u_k = \dfrac{\Delta}{E_k + \xi_k} v_k$; now $u_k^2 + |v_k|^2 = 1$

$\Rightarrow \quad u_k^2 \left[1 + \dfrac{|\Delta|^2}{(E_k + \xi_k)^2} \right] = 1$ or

$$u_k^2 = \frac{1}{1 + \frac{|\Delta|^2}{(E_k + \xi_k)^2}} = \frac{(E_k + \xi_k)^2}{E_k^2 + 2E_k\xi_k + |\Delta|^2} = \frac{(E_k + \xi_k)^2}{2E_k(E_k + \xi_k)}$$

or $u_k^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right)$; as we found in class

$$v_k^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right) \quad - \text{ for } \Delta \text{ chosen real}$$

now $\quad \Delta(r) = V \sum_n v_n^*(r)\, u_n(r)\, [1 - 2f(E_n)]$

$\Rightarrow$

$$\Delta = V \sum_k v_k^*(r)\, u_k(r)\, [1 - 2f(E_k)]$$

$$\Delta = V \sum_k \frac{e^{i(k-k)\cdot r}}{2} \left(1 + \frac{\xi_k}{E_k}\right)^{1/2} \left(1 - \frac{\xi_k}{E_k}\right)^{1/2} [1 - 2f(E_k)]$$

$$\Delta = \frac{V}{2} \sum_k \left(1 - \frac{\xi_k^2}{E_k^2}\right)^{1/2} [1 - 2f(E_k)]$$

$$\Delta = \frac{V}{2} \sum_k \frac{\Delta}{|E_k|} [1 - 2f(E_k)]$$

agrees with class provided $f(E_k) = f(|E_k|)$

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b) Now $\Delta(r) = \Delta e^{iq\cdot r}$

$$\begin{pmatrix} -\frac{\hbar^2\nabla^2}{2m} - \mu & \Delta e^{iq\cdot r} \\ \Delta^* e^{-iq\cdot r} & \frac{\hbar^2\nabla^2}{2m} + \mu \end{pmatrix} \begin{pmatrix} u_n(r) \\ v_n(r) \end{pmatrix} = E_n \begin{pmatrix} u_n(r) \\ v_n(r) \end{pmatrix}$$

let $\quad \begin{bmatrix} u_n(r) \\ v_n(r) \end{bmatrix} = \begin{bmatrix} e^{iq\cdot r/2}\, \tilde{u}_n(r) \\ e^{-iq\cdot r/2}\, \tilde{v}_n(r) \end{bmatrix}$

then get

$$\begin{bmatrix} \left(-\frac{\hbar^2\nabla^2}{2m}-\mu\right)e^{iq\cdot r/2} & \Delta e^{iq\cdot r/2} \\ \Delta^+ e^{-iq\cdot r/2} & \left(\frac{\hbar^2}{2m}\nabla^2+\mu\right)e^{-iq\cdot r/2} \end{bmatrix}\begin{pmatrix} \tilde{u}_n(r) \\ \tilde{v}_n(r) \end{pmatrix} = E_n\begin{pmatrix} e^{iq\cdot r/2}\,\tilde{u}_n(r) \\ e^{-iq\cdot r/2}\,\tilde{v}_n(r) \end{pmatrix}$$

or

$$\begin{bmatrix} e^{-iq\cdot r/2}\left(-\frac{\hbar^2\nabla^2}{2m}-\mu\right)e^{\frac{iq\cdot r}{2}} & \Delta \\ \Delta^+ & e^{iq\cdot r/2}\left(\frac{\hbar^2\nabla^2}{2m}+\mu\right)e^{-iq\cdot r/2} \end{bmatrix}\begin{pmatrix} \tilde{u}_n(r) \\ \tilde{v}_n(r) \end{pmatrix} = E_n\begin{pmatrix} \tilde{u}_n(r) \\ \tilde{v}_n(r) \end{pmatrix}$$

now

$$\nabla^2 e^{i\frac{q\cdot r}{2}} = \nabla\left(e^{i\frac{q\cdot r}{2}}\right)\left[\frac{iq}{2}+\nabla\right] = e^{i\frac{q\cdot r}{2}}\left[\frac{iq}{2}+\nabla\right]^2$$

so we have

$$\begin{bmatrix} -\frac{\hbar^2}{2m}\left[\frac{iq}{2}+\nabla\right]^2-\mu & \Delta \\ \Delta^+ & \frac{\hbar^2}{2m}\left[-\frac{iq}{2}+\nabla\right]^2+\mu \end{bmatrix}\begin{pmatrix} \tilde{u}_n(r) \\ \tilde{v}_n(r) \end{pmatrix} = E_n\begin{pmatrix} \tilde{u}_n(r) \\ \tilde{v}_n(r) \end{pmatrix}$$

this is translationally invariant

$\Rightarrow$ let $\begin{pmatrix} \tilde{u}_n(r) \\ \tilde{v}_n(r) \end{pmatrix} = \begin{pmatrix} u_k \\ v_k \end{pmatrix} e^{ik\cdot r}$, then we get

$$\begin{bmatrix} \frac{\hbar^2}{2m}\left[k+\frac{q}{2}\right]^2-\mu & \Delta \\ \Delta^+ & -\frac{\hbar^2}{2m}\left[k-\frac{q}{2}\right]^2+\mu \end{bmatrix}\begin{pmatrix} u_k \\ v_k \end{pmatrix} = E_k\begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

in 10 we get

$$\begin{bmatrix} \tilde{\xi} + \frac{\hbar^2}{2m} kq & \Delta \\ \Delta^+ & -\tilde{\xi} + \frac{\hbar^2}{2m} kq \end{bmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} = E_k \begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

where $\tilde{\xi} = \frac{\hbar^2}{2m}\left[k^2 + \left(\frac{q}{2}\right)^2\right] - \mu$

$$\Rightarrow \begin{pmatrix} \tilde{\xi} & \Delta \\ \Delta^+ & -\tilde{\xi} \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \left(E_k - \frac{\hbar^2}{2m} kq\right) \begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

this has the same form as the gap equation in part a)

$$\Rightarrow \boxed{E_k - \frac{\hbar^2}{2m} kq = \pm\sqrt{\tilde{\xi}^2 + |\Delta|^2}}$$ give the E_k

$$\boxed{u_k^2 = \frac{1}{2}\left(1 + \frac{\tilde{\xi}}{E_k - \frac{\hbar^2}{2m} kq}\right) \quad \text{and} \quad v_k^2 = \frac{1}{2}\left(1 - \frac{\tilde{\xi}}{E_k - \frac{\hbar^2}{2m} kq}\right)}$$

gap equation becomes

$$\Delta e^{iq \cdot r} = V \sum_k \frac{e^{iq \cdot r}}{2}\left[1 - \frac{\xi^2}{\left(E_k - \frac{\hbar^2}{2m} kq\right)^2}\right]^{1/2}\left[1 - 2f\left(\left|E_k - \frac{\hbar^2}{2m} kq\right|\right)\right]$$

or $$\boxed{\Delta = \frac{V}{2} \sum_k \frac{\Delta}{\left|E_k - \frac{\hbar^2}{2m} kq\right|}\left[1 - 2f\left(\left|E_k - \frac{\hbar^2}{2m} kq\right|\right)\right]}$$

- Note $E_k = \pm\sqrt{\tilde{\xi}^2 + |\Delta|^2} + \frac{\hbar^2}{2m} kq = 0$ is allowed in principle

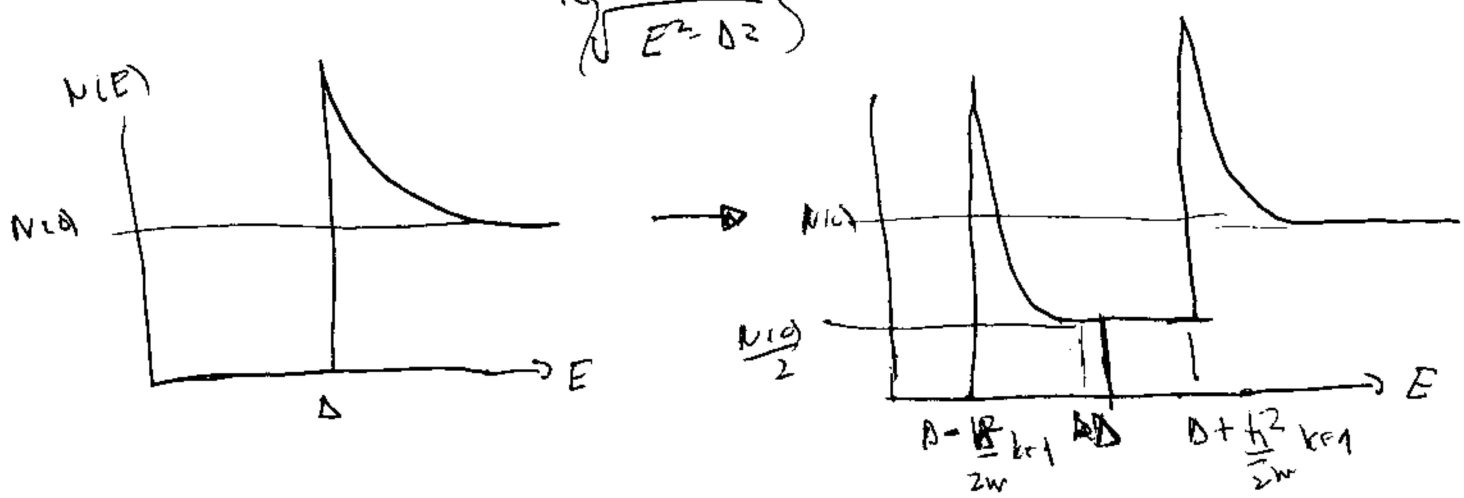
$\Rightarrow$ quasi-particle gap can be removed and $N(E) \to N(E \mp \frac{\hbar^2}{2m} kq)$,
meaning it shift

Density of states:

$$E = \sqrt{\xi^2 + |\Delta|^2} + \frac{\hbar^2}{2m} k q \quad \text{in } \underline{1D} \qquad kq = \pm k_F q$$

$$N(E) \to \frac{N(E_+) + N(E_-)}{2} = \frac{N\left(E + \frac{\hbar^2}{2m} k_F q\right) + N\left(E - \frac{\hbar^2}{2m} k_F q\right)}{2}$$

where $N(E) = N(0)\, \mathrm{Re}\left\{ \frac{E}{\sqrt{E^2 - \Delta^2}} \right\}$



- The density of states splits into two peaks
- The lower peak can move to $E = 0$, so the gap can close.

2 a) The new term is

$$k_2\left[(D_x\Psi_1)(D_y\Psi_2)^* + (D_x\Psi_2)(D_y\Psi_1)^* + c.c.\right]$$

↑ means complex conjugate

$$D_i = \left(-i\hbar\nabla_i - \frac{e}{c}A_i\right)$$

For the δA_x portion, this gives
δA_y portion

$$-\frac{e}{c}k_2\Big[\delta A_x\left(\Psi_1(D_y\Psi_2)^* + \Psi_2(D_y\Psi_1)^* + \Psi_1^*(D_y\Psi_2) + \Psi_2^*(D_y\Psi_1)\right)$$
$$+\delta A_y\left(\Psi_2^*(D_x\Psi_1) + \Psi_1^*(D_x\Psi_2) + \Psi_2(D_x\Psi_1)^* + \Psi_1(D_x\Psi_2)^*\right)\Big]$$

so the expressions for the supercurrents become

$$\boxed{\begin{aligned} j_x &= k_1 e\left[\Psi_1^*(D_x\Psi_1) + \Psi_1(D_x\Psi_1)^* + \Psi_2^*(D_x\Psi_2) + \Psi_2(D_x\Psi_2)^*\right] \\ &\quad + k_2 e\left[\Psi_1(D_y\Psi_2)^* + \Psi_2(D_y\Psi_1)^* + \Psi_1^*(D_y\Psi_2) + \Psi_2^*(D_y\Psi_1)\right] \\ j_y &= k_1 e\left[\Psi_1^*(D_y\Psi_1) + \Psi_1(D_y\Psi_1)^* + \Psi_2^*(D_y\Psi_2) + \Psi_2(D_y\Psi_2)^*\right] \\ &\quad + k_2 e\left[\Psi_2(D_x\Psi_1)^* + \Psi_1(D_x\Psi_2)^* + \Psi_2^*(D_x\Psi_1) + \Psi_1^*(D_x\Psi_2)\right] \end{aligned}}$$

b) $(\psi_1, \psi_2) = \psi_0 \,[\tanh(qx), i]$

want to choose ψ_0 and q so that this solution satisfies the equation for ψ_1.

- to find ψ_0, we know that far from the boundary we must have the bulk solution

$\Rightarrow$ consider the bulk solution (all spatial variations zero)

$$f_H = \alpha(|\psi_1|^2 + |\psi_2|^2) + \beta(|\psi_1|^2 + |\psi_2|^2)^2 + \beta_2(\psi_1^2\psi_2^{*2} + \psi_2^2\psi_1^{*2})$$

let $(\psi_1, \psi_2) = \psi_0(1, i)$, since this will be the bulk solution

$\Rightarrow f_H = 2\alpha\psi_0^2 + 4\beta_1\psi_0^4 - 2\beta_2\psi_0^4$ is FGL.

$$\frac{\partial f_H}{\partial \psi_0^2}: \quad 2\alpha + (8\beta_1 - 4\beta_2)\psi_0^2 = 0$$

$$\Rightarrow \boxed{\psi_0^2 = \frac{-\alpha}{4\beta_1 - 2\beta_2}} \quad \text{this gives } \psi_0$$

- Next we need to find q; include spatial variations in $\psi_1(x)$:

$$f = \alpha|\psi_1|^2 + \beta_1(|\psi_1|^2 + |\psi_2|^2)^2 - \beta_2\psi_0^2(\psi_1^2 + \psi_1^{*2}) + k\left(\frac{\partial\psi_1}{\partial x}\right)^2 + k_2 \cdot 0$$

$$df = \delta\psi_1^* \left[\alpha\psi_1 + 2\beta_1(|\psi_1|^2 + |\psi_2|^2)\psi_1 - 2\beta_2\psi_0^2\psi_1^* - k\frac{\partial^2}{\partial x^2}\psi_1\right] = 0$$

$$\Rightarrow \alpha\psi_1 + 2\beta_1|\psi_1|^3 + (2\beta_1 - 2\beta_2)\psi_0^2\psi_1 - k\frac{d^2}{dx^2}\psi_1 = 0$$

$$\text{or } \left(\alpha + (2\beta_1 - 2\beta_2)\psi_0^2 - k\frac{d^2}{dx^2}\right)\psi_1 + 2\beta_1\psi_1^3 = 0$$

$$\text{or } \left(\alpha - \frac{\alpha(2\beta_1 - 2\beta_2 + 2\beta_1)}{4\beta_1 - 2\beta_2} + \frac{\alpha\, 2\beta_1}{4\beta_1 - 2\beta_2} - k\frac{d^2}{dx^2}\right)\psi_1 + 2\beta_1\psi_1^3 = 0$$

or $$\boxed{\left(-2B_1\psi_0^2 - k\frac{d^2}{dx^2}\right)\psi_1 + 2B_1\psi_1^3 = 0}$$ is the equation for $\psi_1(x)$

let $$\psi_1 = \psi_0 \tanh qx$$

$$\frac{d}{dx}\tanh qx = q\,\mathrm{sech}^2 qx$$

$$q\frac{d}{dx}\mathrm{sech}^2 qx = q\;2\,\mathrm{sech}\,qx\,\frac{d}{dx}\mathrm{sech}\,qx$$

$$= -2q^2\,\mathrm{sech}^2 qx\,\tanh qx$$, so our equation

becomes

$$\psi_0\left(-2B_1\psi_0^2 + 2q^2\,\mathrm{sech}^2 qx\right)\tanh qx + 2B_1\psi_0^3\tanh^3 qx \overset{?}{=} 0$$

now $\tanh^2 qx = 1 - \mathrm{sech}^2 qx$

$$\Rightarrow \left(-2B_1\cancel{\psi_0^2} + k2q^2\,\mathrm{sech}^2 qx\right) + 2B_1\cancel{\psi_0^2}\left(\cancel{1} - \mathrm{sech}^2 qx\right) \overset{?}{=} 0$$

$$\Rightarrow \left(k2q^2 - 2B_1\psi_0^2\right)\mathrm{sech}^2 qx \overset{?}{=} 0$$

possible if $\boxed{q^2 = \frac{B_1\psi_0^2}{k}}$

$$\Rightarrow \boxed{\psi_1(x) = \psi_0 \tanh\left(\frac{B_1\psi_0^2 x}{k}\right)}$$ is the solution for $\psi_1(x)$

the satisfies the G.L. equation, this $q^2 \sim \frac{1}{\xi^2(T)}$ case expected

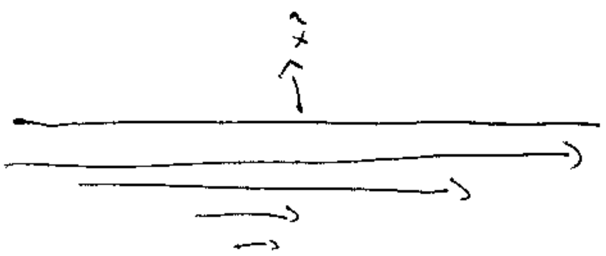
but $q^2 \neq \frac{1}{\xi^2(T)}$ in general

$$j_x = k_1 e \psi_0^2 \left[\tanh qx \, i \frac{d}{dx}(\tanh qx) - i \tan qx \frac{d}{dx}(\tanh qx) \right] = 0$$

$$j_y = k_2 e \psi_0^2 \left[i \left(+ i \frac{d}{dx} \tanh x \right) - i \left(-i \frac{d}{dx} \tanh x \right) \right]$$

$$j_y = -2 e k_2 \psi_0^2 \frac{d}{dx}(\tanh x) = -2 e k_2 \psi_0^2 q \operatorname{sech}^2 qx$$

$$\boxed{j_y = -2e k_2 \psi_0^2 q \operatorname{sech}^2 qx}$$



j_y decays a length $\frac{1}{q}$ in the superconductor from the surface.