

Sect 2.1 Dynamics of Mechanical Systems

- Systems with storage introduce differential equations:

$$\frac{d \text{Stored Quantity}}{dt} = \text{Force, or Rate of Flow}$$

Examples:

- $E = \frac{1}{2} C v^2 \longrightarrow \text{Current} = \text{Capacitance} * \frac{d}{dt} \{ \dots \dots \dots \}$
- $E = \frac{1}{2} M v^2 \longrightarrow \text{Force} = \text{Mass} * \frac{d}{dt} \text{Velocity}$
- $E = C_q$

Storage is often energy storage, but also business systems (inventory), economics (accounts), etc.

Why does the Controls Engineer need Dynamics ?

- A model of the thing to be controlled is needed for $\{ \dots \dots \dots \}$

How ?

- From (for linear motion)

$$F(t) = mA(t) \quad \left(F(t) = m \frac{d^2 x(t)}{dt^2} \right)$$

- Or (for $\{ \dots \dots \dots \}$ motion)

$$M(t) = I \ddot{\Theta}(t)$$

Procedure for electro-mechanical systems

1. { {
 - (a) Draw of schematic diagram
 - (b) Assign variables (x or θ) necessary to determine the { { of all moving masses
 - (c) List signals and parameters and model equations
2. Draw a free-body diagram of each component
 - Indicate all forces acting on each body
3. Apply { { third law. (All forces occur in equal but oppositely directed pairs).
4. Combine equations to eliminate internal forces.
5. Apply Newton's second law, { {

Example 2.1 Cruise Control

Schematic:

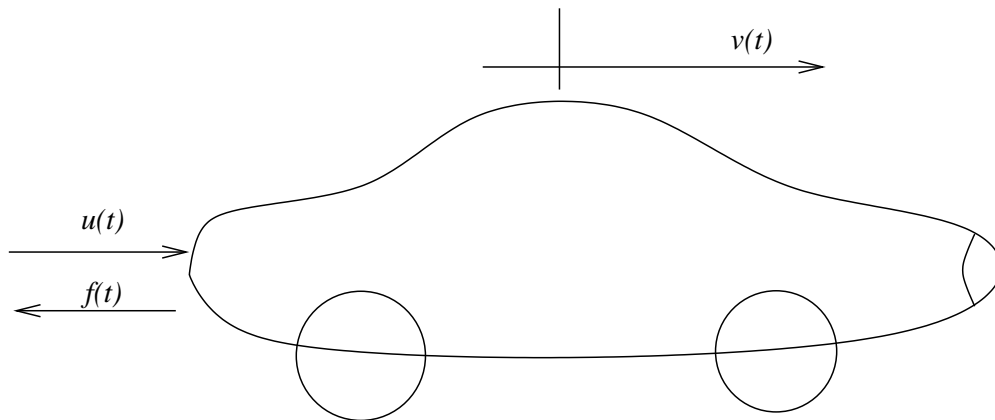


Figure 1: Schematic

Signals:

Signal	Units	Name
$u(t)$	[N]	Applied Force
$v(t)$	[m/s]	Car Velocity
$f(t)$	[N]	Drag Force

Table 1: Signals.

Parameters:

Parameter	Value	Units
Mass	M	[Kg]
Drag Coefficient	b	[N/m/s]

Table 2: Parameters.

Example 2.2 Car Suspension

Schematic

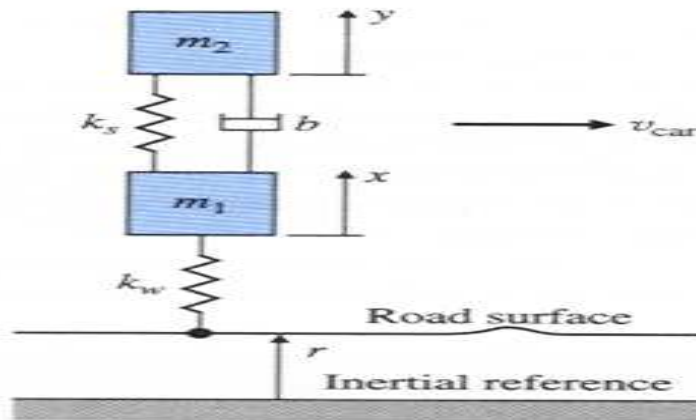


Figure 3: Car Suspension, schematic.

Signal	Units	Function
$x(t)$	[m]	Position of Tire Center
$y(t)$	[m]	Position of Car Frame
$r(t)$	[m]	Road Position at instant t

Parameter	Value	Units
Tire Stiffness	k_w	[N/m]
Spring Stiffness	k_s	[N/m]
Shock Absorber	b	[N-s/m]
Mass of Tire	M_1	[Kg]
$\frac{1}{4}$ Mass of Car	M_2	[Kg]

Free-body diagram

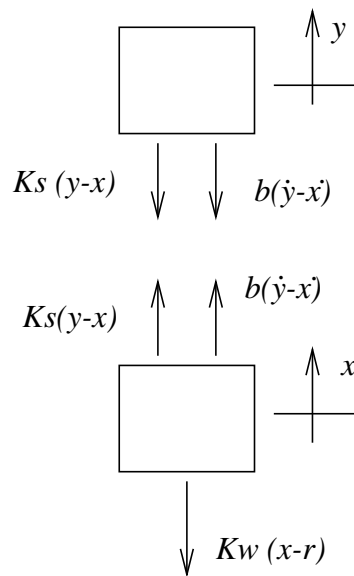


Figure 4: Free-body diagram.

Newton's Second Law

$$\text{Mass 1 : } b(\dot{y} - \dot{x}) + k_s(y - x) - k_w(x - r) = M_1\ddot{x}$$

$$\text{Mass 2 : } -b(\dot{y} - \dot{x}) - k_s(y - x) = M_2\ddot{y}$$

Next: { **Equations**

- Possibilities:
 - Find $x(t)$ in terms of $r(t)$
 - Find $y(t)$ in terms of $r(t)$
- Much easier with { (will do later).

Example 2.3 Rotational Motion: Satellite Control

Schematic

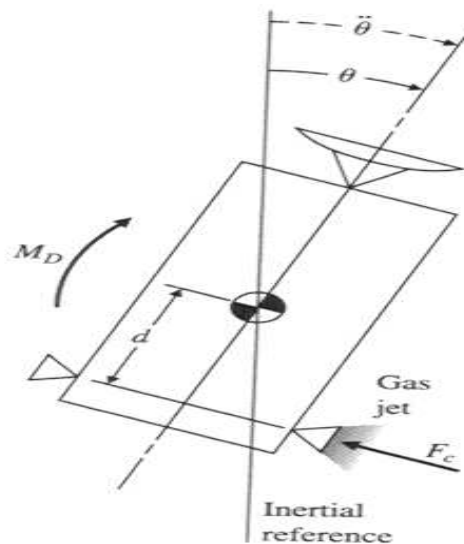


Figure 5: Satellite Control, schematic.

Signal	Units	Function
$\Theta(t)$	[radians]	Orientation of satellite
$F_c(t)$	[N]	Thrust (force)
$M_D(t)$	[N-m]	Disturbance (torque)

Table 3: Signals.

Parameter	Value	Units
Inertia	I	[Kg-m ²]
Thruster radius	b	[m]

Table 4: Parameters.

Free-body diagram

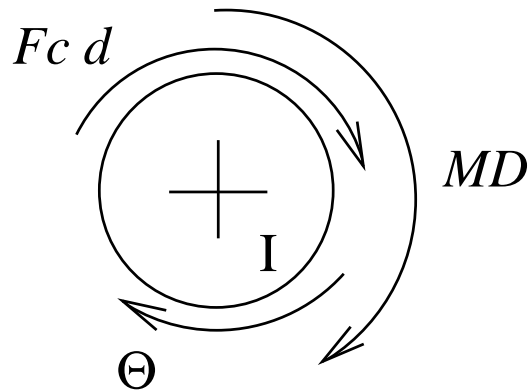


Figure 6: Free-body diagram.

Newton's Second Law

$$F_c(t) d + M_D(t) = I \ddot{\Theta}(t) \quad (1)$$

Combine Equations:

No equations to combine in this case. Write in Standard form:

$$\ddot{\Theta}(t) = F_c(t) d + M_D(t) \quad (2)$$

Example 2.4 Flexibility: Flexible Read/write head for a Disk Drive

Schematic

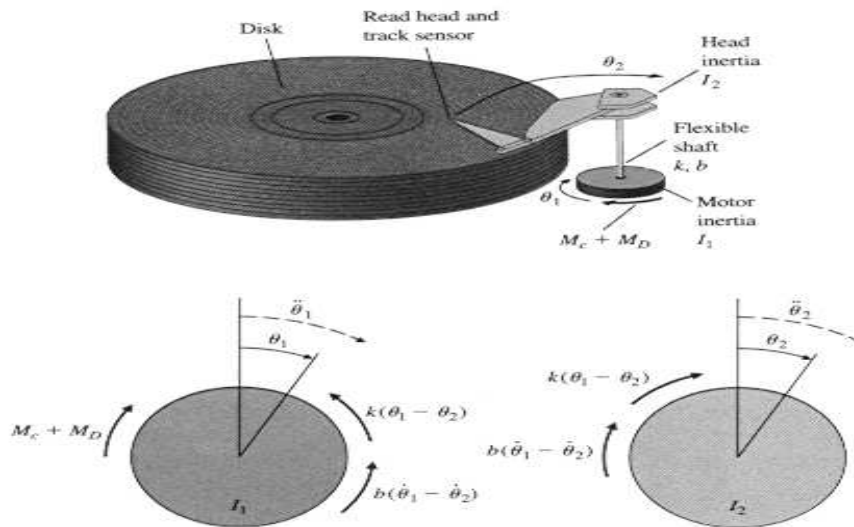


Figure 7: Disk Head Motion, schematic and free body diagram.

Signal	Units	Function
$\Theta_1(t)$	[rad]	Position of motor
$\theta_2(t)$	[rad]	Position of disk head
$M_c(t)$	[N-m]	{
$M_D(t)$	[N-m]	Disturbance Torque

Table 5: Signals.

Parameter	Value	Units
Motor Inertia	I_1	[Kg-m ²]
Head Inertia	I_2	[Kg-m ²]
Shaft Stiffness	k	[N-m/rad]
Shaft Damping	b	[N-m-s/rad]

Table 6: Parameters.

Newton's Laws

$$\begin{aligned}I_1\ddot{\theta}_1 &= M_c + M_D - k(\theta_1 - \theta_2) - b(\dot{\theta}_1 - \dot{\theta}_2) \\I_2\ddot{\theta}_2 &= k(\theta_1 - \theta_2) + b(\dot{\theta}_1 - \dot{\theta}_2)\end{aligned}$$

or rewrite:

$$\begin{aligned}I_1\ddot{\theta}_1 + b(\dot{\theta}_1 - \dot{\theta}_2) + k(\theta_1 - \theta_2) &= M_c + M_D \\I_2\ddot{\theta}_2 + b(\dot{\theta}_2 - \dot{\theta}_1) + k(\theta_2 - \theta_1) &= 0\end{aligned}$$

Combine Equations:

This is a piece of work. We will see an easier way using State Space Modeling.

Read examples 2.5 and 2.6

Look at problems 2.1, 2.2 and 2.4

1 Procedure to Develop The Analogous Electrical System

1. Make a list of signals and parameters of the Mechanical / Fluid / Thermal System. For each parameter or signal, determine the electrical analog.
2. Identify which of the Mechanical / Fluid / Thermal variables are across variables (corresponding to voltages), and which are through variables (corresponding to currents).
3. Draw electrical circuit nodes (dots), 1 corresponding to each across variable in the Mechanical / Fluid / Thermal System.
4. For each Mechanical / Fluid / Thermal component, put in the analogous electrical component:
 - (a) Resistors, Inductors: between electrical nodes corresponding to across variables of endpoints in Mechanical / Fluid / Thermal System.
 - (b) Capacitors: between one electrical node and ground; electrical node corresponding to the across variable of the Mass, Inertia or Fluid or Thermal capacitor.
5. A Force / Torque / Flow / Flux input corresponds to a Current source in the circuit.
6. A Velocity / Angular Velocity / Pressure / Temperature input corresponds to a Voltage source in the circuit.
7. Position signal must be converted to velocity signals by integrating, represented by (1/s) in the transfer function: for example:

$$\frac{X(s)}{F(s)} = \frac{1}{s} \frac{\text{Vel}(s)}{F(s)} \Leftrightarrow \frac{V(s)}{I(s)}$$

8. Analyze the circuit to determine the desired electrical transfer function; simplify as much as possible.
9. Verify the correctness of units in the transfer function.
10. Replace the terms of the electrical transfer function with the corresponding Mechanical / Fluid / Thermal terms and simplify.
11. Verify the correctness of units in the transfer function.

Dynamic Modeling by Analogy to Electrical Circuits: Variables and Differential Equations

System Type	Across Variable	Through Variable	Energy Storage		Energy Dissipation
			Capacitor-Like	Inductor-Like	Resistor-Like
Electrical	Voltage, V [volts]	Current, I [amps]	Capacitor, C $I = C\dot{V}$ $E = \frac{1}{2}CV^2$ [Joules]	Inductor, L $(V_1 - V_2) = L\dot{I}$ $E = \frac{1}{2}LI^2$ [Joules]	Resistor, R $(V_1 - V_2) = IR$ $P = VI = V^2/R = I^2R$ [W]
Mechanical (rotational units)	Velocity, $v = \dot{x}$ (ω) [meters/sec.] (radians/sec)	Force (Torque), F (T) [newtons] (newton-meters)	Mass (Inertia), M (J) $F = M\ddot{x}$ $T = J\ddot{\theta} = J\dot{\omega}$ $E = \frac{1}{2}Mv^2$ ($\frac{1}{2}J\omega^2$) [J]	Spring, k $F = kx$ ($\dot{x} = 1/k\dot{F}$) $T = k\theta$ ($\dot{\omega} = 1/k\dot{T}$) $E = \frac{1}{2}\frac{1}{k}F^2 = \frac{1}{2}kx^2$ [J]	Damper, b $F = b(\dot{x} - \dot{x}_2)$ $T = b(\omega_1 - \omega_2)$ $P = vF = v^2b = F^2/b$ [W]
Fluid	Pressure, p [N/meter ²]	Flow, q, W (See Note 3)	Fluid Compressibility, C_h, C_g $q = C_h \dot{p}$ $W = C_g \dot{p}$	Fluid Inertia, I_f $(p_1 - p_2) = I_f \dot{q}$	Fluid Resistance (to fluid flow) $(p_1 - p_2) = qR_f$
Thermal	Temperature, T [°C]	Heat Flux, q [watts]	Thermal Capacitance, C_t $q = C_t \dot{T}$	NONE (except near absolute 0°)	Thermal Resistance (to heat flow) $(T_1 - T_2) = qR_t$

- Across Variables:** Note that the Energy Dissipation equations relate voltage/velocity/pressure/temperature **difference** to flow. This is because voltage/velocity/pressure/temperature are across variables.
- The direction of through variable measurement must be related to the direction of across variable measurement by the passive sign convention (arrow in at the '+').
- For fluids, two flows are considered:
 - Volume flow rate, q [meters³/sec].
 - Mass flow rate, $W = q \rho_o$ [Kg/sec], where ρ_o is the fluid density [Kg/meters³].
- The electrical analogs of capacitors in Mechanical/Fluid/Thermal systems are always capacitors between a node and ground

Units

System Type	Resistor	Capacitor	Inductor
Across/Through	Resistance is: Across/Through	Capacitance is: Through / (Across/sec)	Inductance is: Across / (Through/sec)
Electrical	R : Volts/Amp [Ohms] = [Volts/Amp]	C : Amps / (Volt/second) [Farad] = [A-s/V]	L : Volts / (Amp/sec) [Henry] = [V - s/A]
Mechanical, (linear)	b : Force / Velocity [N-s/m] = [Kg/s] $R \leftrightarrow 1/b$	M : Force / Acceleration [N / (m/s ²)] = [N - s ² /m] = [Kg] $C \leftrightarrow M$	k : Force / Position 1 / (Velocity / (Force/sec)) $L \leftrightarrow 1/k$
Mechanical, (rotational)	b : Torque / Velocity [N-m-s/rad] = [Kg-m ² /rad-s] $R \leftrightarrow 1/b$	J : Torque / Acceleration [(N - m) / (rad/s ²)] = [Kg - m ²] $C \leftrightarrow J$	k : Torque / Angle 1 / (Velocity / (Torque/sec)) $L \leftrightarrow 1/k$
Fluid	R_f : Pressure / Flow [(N/m ²) / (m ² /s)] = [N - s/m ⁵] $R \leftrightarrow R_f$	C_h, C_g : Flow / (Pressure/sec) C_h : [m ⁵ /s ² - N] C_g : [Kg - m ² /s ² - N] $C \leftrightarrow C_h, C_g$	I_f : Pressure / (Flow/sec) [(N/m ²) / (m ³ /s ²)] = [N - s ² /m ⁵] $L \leftrightarrow I_f$
Thermal	R_t : Degrees / Heat Flux [°C/watt] $R \leftrightarrow R_t$	C_t : Heat Flux / (Deg/sec) [Watt - s/°C] = [Joules/°C] $C \leftrightarrow C_t$	NONE
1. V: volts, A: amps. 2. N: newtons, S: seconds, m: meters, rad: radians. 3. Force: 1 [N] = 1 [Kg-m/s ²]; Work: 1 [Joule] = 1 [N-m]; Power: 1 [Watt] = 1 [N-m/s].			